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Examples 08

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-08.pdf]

For feedback on these examples, please get your write-ups to me by Friday March 9, 2018.

[08.1] Recall the proof of the spectral theorem for self-adjoint operators on a finite-dimensional complex vector space V with hermitian inner product. Recall the proof of a spectral theorem for *two* self-adjoint operators S, T on V under the assumption that $ST = TS$.

[08.2] Let $K(x, y) = |x - y|$, and let

$$Tf(x) \;=\; \int_a^b K(x,y)\; f(y)\; dy \qquad\qquad (\text{for } f \in L^2[a,b])$$

Find some eigenvalues/eigenfunctions for the operator T. (*Hint*: consider $\frac{d^2}{dx^2}(Tf)$ and use the fundamental theorem of calculus.)

[08.3] Let $K(x, y) \in L^2([a, b] \times [a, b])$, and attempt to define a map $T : L^2[a, b] \to L^2[a, b]$ by

$$Tf(x) \;=\; \int_a^b K(x,y)\, f(y)\; dy$$

Show that Tf is well-defined a.e. as a pointwise-valued function. Show that T really does map L^2 to itself by showing that

$$|Tf|_{L^2[a,b]} \;\le\; |K|_{L^2([a,b]\times[a,b])} \cdot |f|_{L^2[a,b]}$$

(One would say that $K(,)$ is a *Schwartz kernel* for the map T. Yes, this use is in conflict with the use of *kernel* of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint T^* of T has Schwartz kernel $\overline{K(y,x)}$.

[08.4] Prove that the *Volterra operator* $Vf(x) = \displaystyle\int_0^x f(t)\; dt$ on $C^o[0, 1]$ or on $L^2[0, 1]$ has no (not-identically-zero) eigenvalues/eigenvectors.

[08.5] Determine the spectrum of the *left-shift* $L : (c_1, c_2, \ldots) \to (c_2, \ldots)$ on ℓ^2, and of the *right-shift* $R : (c_1, c_2, \ldots) \to (0, c_1, c_2, \ldots)$ on ℓ^2. Show that these are mutual *adjoints*.

[08.6] The Volterra operator $Vf(x) = \int_0^x f(y)\, dy$ on $L^2[0, 1]$ has kernel

$$K(x,y) \;=\; \begin{cases} 1 & (\text{for } 0 \le y \le x \le 1) \\ \\ 0 & (\text{for } 0 \le x < y \le 1) \end{cases}$$

Determine the (Schwartz) kernel for $T = V \circ V^*$. Find some eigenfunctions for T. (Recall that V has no eigenfunctions!) (*Hint:* apply d/dx to the equation $Tf = \lambda \cdot f$ and presume that the differentiation passes inside the integral.)

[08.7] *(Approximate eigenvectors and continuous spectrum: Weyl's criterion)* Let $T : V \to V$ be a self-adjoint linear operator on a Hilbert space V. For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1) such that $(T - \lambda)v_n \to 0$ as $n \to +\infty$ is an *approximate eigenvector* for λ. Show that for λ *not* an eigenvalue for T, λ has an approximate eigenvector if and only if λ is in the spectrum of T.

[08.8] Show that the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Tf(x) = f(x) \cdot \sin x$ has empty discrete spectrum. Show that it is self-adjoint. Show that T has continuous spectrum the interval $[-1, 1]$. (We know that self-adjoint (or merely *normal*) operators have only point spectum and continuous spectrum, that is, no left-over *residual* spectrum.)

[08.9] Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers inside the interval $[0, 1]$. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the $r_1, r_2, \ldots$, and continuous spectrum the whole interval $[0, 1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).
