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Examples 09

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-09.pdf]

For feedback on these examples, please get your write-ups to me by Friday March 23, 2018.

[09.1] Show that the translation action $T_x f(y) = f(y+x)$ on the Banach space $C^o_{\rm bdd}(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$, with sup norm, is *not* continuous. That is, $\mathbb{R}\times C^o_{\rm bdd}(\mathbb{R}) \to C^o_{\rm bdd}(\mathbb{R})$ by $x\times f \to T_x f$ is *not* continuous. In particular, find a particular $f \in C^o_{\rm bdd}(\mathbb{R})$ with $|f|_{C^o} = 1$ such that, there is a sequence $\delta_n \to 0$ of non-zero numbers δ_n such that $|T_{\delta_n} f - f|_{C^o} = 1$.

[09.2] Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers inside the interval $[0,1]$. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the $r_1, r_2, \ldots$, and continuous spectrum the whole interval $[0,1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

[09.3] Let $r_1, r_2, r_3, \ldots$ be a bounded sequence of complex numbers. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is *compact* if and only if $r_n \to 0$.

[09.4] Let T be a compact operator $T : V \to W$ for Hilbert spaces V, W. For S a continuous/bounded operator on V, show that $T \circ S : V \to W$ is compact. For R a continuous/bounded operator on W, show that $R \circ T : V \to W$ is compact.

[09.5] Let S, T be two compact, self-adjoint operators on a Hilbert space, and $ST = TS$. Show that there is an orthonormal basis for V consisting of simultaneous eigenfunctions for S, T.

[09.6] Recall the proof that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals r_n,

$$C(r) \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le r_n\}$$

is compact if and only if $\sum_n |r_n|^2 < \infty$.

[09.7] First, for Schwartz φ on $\mathbb{R}^n$ and u a tempered distribution on $\mathbb{R}^n$, characterize $\varphi * u$. Show that $\widehat{\varphi * u} = \widehat{\varphi} \cdot \widehat{u}$, where the latter multiplication is that induced by duality: $(\widehat{\varphi} \cdot \widehat{u})(\psi) = \widehat{u}(\widehat{\varphi} \cdot \psi)$ for $\psi \in \mathscr{S}$. Explain why the union $H^{-\infty}$ of Sobolev spaces is inside the space of tempered distributions, and why $\widehat{u}$ has pointwise values for $u \in H^{-\infty}$.
