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Examples 10

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-10.pdf]

For feedback on these examples, please get your write-up to me by Friday, April 20.

[10.1] For a bounded sequence $\lambda_1, \lambda_2, \ldots$ of complex numbers, let $T : \ell^2 \to \ell^2$ by let $T(z_1, z_2, \ldots) = (\lambda_1 z_1, \lambda_2 z_2, \ldots)$. Show that the whole spectrum of T is the (topological) closure of the set $\{\lambda_1, \lambda_2, \ldots\}$.

[10.2] Let $T : \ell_{\mathbb{Z}} \to \ell_{\mathbb{Z}}$ be the *two-sided right-shift* on $\ell^2_{\mathbb{Z}}$, given by $(Tz)_n = z_{n-1}$, where $z = (\ldots, z_{-1}, z_0, z_1, z_2, \ldots) \in \ell^2_{\mathbb{Z}}$. Observe that T is *normal*, in the sense that $TT^* = T^*T$, and that the adjoint T^* is the two-sided *left*-shift. Determine the eigenvalues and the whole spectrum of T.

[10.3] Let $K(,)$ be a measurable function on $\mathbb{R}^2$, with a bound B such that $\int_{\mathbb{R}} |K(x,y)|\,dx \le B$ for every y, and $\int_{\mathbb{R}} |K(x,y)|\,dy \le B$ for every x. Show that $Tf(x) = \int_{\mathbb{R}} K(x,y)\,f(y)\,dy$ gives a continuous linear map $L^p \to L^p$ for every $1 < p < \infty$, with $|Tf|_{L^p} \le B \cdot |f|_{L^p}$. (*Hint:* Hölder's inequality.)

[10.4] **Simple instance of** *Young's inequality:* In the previous example, let $K(x,y) = k(x-y)$ for $k \in L^1(\mathbb{R})$, so that $Tf(x) = (k * f)(x)$. Show that $|Tf|_{L^p} \le |k|_{L^1} \cdot |f|_{L^p}$.

[10.5] Write the Fourier series for the Schwartz kernel for

[10.6] Write the Fourier series for the Schwartz kernel for the identity map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$, and show that it is in $H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^{2n})$ for every $\varepsilon > 0$.

[10.7] **Cartan-Eilenberg adjunction:** For abelian groups A, B, C, prove that $\mathrm{Hom}(A, \mathrm{Hom}(B,C)) \approx \mathrm{Hom}(A \otimes B, C)$.

[10.8] $V \otimes V^* \to \mathrm{End}_k(V)$ for finite-dimensional k-vector-spaces V, over a field k. Let V^* be its k-linear dual, $\mathrm{Hom}_k(V,k)$. Show that the linear map $V \otimes V^* \to \mathrm{End}_k(V)$ induced from the map $V \times V^* \to \mathrm{End}_k(V)$ given by

$$v \times \lambda \longrightarrow \Big(w \longrightarrow \lambda(w) \cdot v\Big) \qquad \text{(with } v, w \in V \text{ and } \lambda \in V^*)$$

gives an *isomorphism* $V \otimes V^* \to \mathrm{End}_k(V)$.

[10.9] **Coordinate-independent expression for** *trace***:** With finite-dimensional k-vector-space V, let $\tau : V \times_k V^* \to k$ be the natural bilinear pairing $v \times \lambda \to \lambda(v)$. Show that this map, composed with the previous, induces the *trace* map on $\mathrm{End}_k(V)$.

[10.10] Let V be a Hilbert space. Show that the *algebra* tensor product $V \otimes_{\mathrm{alg}} V^*$ is naturally isomorphic to the *finite-rank* operators $V \to V$, with the isomorphism uniquely and completely specified by $v \otimes \lambda \to (w \to \lambda(w) \cdot v)$.

[10.11] **Impossibility of extending trace from finite-rank operators to Hilbert-Schmidt operators:** Let A be the finite-rank operators on an infinite-dimensional Hilbert space V (for example, ℓ^2), identified with $V \otimes_{\mathrm{alg}} V^*$. Let $|\cdot|_{\mathrm{HS}}$ be the Hilbert-Schmidt norm, specified by $|v \otimes \lambda|_{\mathrm{HS}} = |v| \cdot |\lambda|$. Show that *trace* on A does *not* extend continuously to Hilbert-Schmidt operators. Explain how this implies that trace does not extend continuously to the collection of *all* continuous linear operators (with operator norm).

[10.12] For *Hermite polynomials* $H_n(x)$ defined by $H_n(x) = e^{x^2} \cdot \frac{d^n}{dx^n} e^{-x^2}$, show that the various H_n's are *orthogonal* in the weighted L^2 space $L^2(\mathbb{R}, e^{-x^2}dx)$, that is, with measure e^{-x^2} times the usual Lebesgue measure on $\mathbb{R}$.
