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Review examples discussion 01

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2017-18/real-disc-01.pdf]

[01.1] Show that the closed unit ball in ℓ^2, although *closed and bounded*, is *not compact*, by showing it is not *sequentially compact*.

Discussion: Let $e_n = (0,\ldots,0,1,0,\ldots)$ with the single 1 at the n^{th} place. Then $d(e_m, e_n) = \sqrt{2}$ for $m \neq n$. Thus, the sequence of e_n's has no Cauchy subsequence, so no convergent subsequence. ///

[01.2] Show that the closed unit ball in $C^o[a,b]$ is not compact, despite being closed and bounded.

Discussion: Let f_n be a tent function centered at $1/2^n$, of height 1, and width $1/2^{n+2}$ (or anything strictly larger than $1/2^{n+1}$). By design, the supports of these functions are disjoint, and all their sup-norms are 1. Thus, for $m \neq n$, $|f_m - f_n|_{C^o} = 1$. Thus, the sequence has no Cauchy subsequence. ///

[01.3] Let X be a metric space with a countable dense subset D. Show that every open set in X is a countable union of open balls.

Discussion: Let U be the open set. For $x \in U$, let $B(r_x, x)$ be an open ball or radius r_x centered at x and contained in U. We can shrink r_x to make it rational. By density, there is an element d_x in the smaller ball $B(r_x/2, x)$. Then $B(r_x/2, d_x)$ contains x and is inside $B(r_x, x)$, so is inside U. Thus, $U \subset \bigcup_{x\in U} B(r_x/2, d_x)$. By countability of D and of rationals (the radii), there can be only countably-many distinct balls $B(r_x, d_x)$. ///

[01.4] Let X be a compact metric space. Show that a continuous function on X is *uniformly* continuous.

Discussion: Let $f \in C^o(X)$. Given $\varepsilon > 0$, for each $x \in X$ let $B(r_x, x)$ be a ball of radius r_x centered at x such that $|f(x) = f(y)| < \varepsilon$ for $y \in B(r_x, x)$. The open sets $B(r_x/2, x)$ cover X. By compactness, there is a finite subcover $B(r_{x_1}/2, x_1), \ldots, B(r_{x_n}/2, x_n)$. Thus, given $y, z \in X$ with $d(y,z) < \min_i r_{x_i}/2$, let $y \in B(r_{x_i}/2, x_i)$. Then $z \in B(r_{x_i}, x_i)$, as is y. Thus, $|f(y) - f(z)| < \varepsilon$. ///

[01.5] Let X be a compact metric space. Show that a uniform pointwise limit of continuous real-valued functions is continuous.

Discussion: This is a slightly abstracted version of the iconic three-epsilon argument. Let $\{f_n\}$ be a *uniformly* pointwise convergent sequence of continuous functions on X. In particular, it is pointwise convergent at every $x \in X$, so it has a pointwise limit $f(x) = \lim_n f(x)$ for each x. We claim that $f(x)$ is continuous. Given $\varepsilon > 0$, choose n_o sufficiently large so that for $m, n \geq n_o$ and for all $x \in X$ we have $|f_m(x) - f_n(x)| < \varepsilon$. This implies that $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in X$ and $n \geq n_o$. Fix $x_o \in X$. Let $\delta > 0$ be such that for $d(x_o, y) < \delta$ we have $|f_{n_o}(x_o) - f_{n_o}(y)| < \varepsilon$. Then

$$|f(x_o)-f(y_o)| \;\leq\; |f(x_o)-f_{n_o}(x_o)|+|f_{n_o}(x_o)-f_{n_o}(y)|+|f_{n_o}(y)-f(y)| \;\leq\; \varepsilon+|f_{n_o}(x_o)-f_{n_o}(y)|+\varepsilon \;<\; \varepsilon+\varepsilon+\varepsilon$$

proving continuity. ///

[01.6] Show that $C^o[a,b]$ is not complete with the $L^2[a,b]$ metric.

Discussion: That is, we want a sequence $\{f_n\}$ of C^o functions that is Cauchy in the L^2 metric, but not in the C^o metric. In particular, it would suffice to find $\{f_n\}$ which converge in L^2 to an L^2 function which is not C^o.

For example, $\{f_n\}$ can be a sequence of continuous, piecewise-linear functions converging pointwise to a step function (which is certainly not continuous). For example, with $[a, b] = [0, 1]$,

$$f_n(x) \;=\; \begin{cases} 0 & (\text{for } 0 \le x < \frac{1}{2} - \frac{1}{n}) \\ \frac{n}{2} \cdot (x - \frac{1}{2} + \frac{1}{n}) & (\text{for } \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n}) \\ 1 & (\text{for } \frac{1}{2} + \frac{1}{n} < x \le 1) \end{cases}$$

The graph is flat to the left and flat to the right, and has a straight line of slope $n/2$ connecting the two flat parts. The pointwise limit is a step function with step of height 1 at $\frac{1}{2}$.

For $m \le n$ the L^2 norm of $f_m - f_n$ is easily estimated by

$$|f_m - f_n|^2_{L^2} \;=\; \int_{\frac{1}{2}-\frac{1}{m}}^{\frac{1}{2}+\frac{1}{m}} |f_m(x) - f_n(x)|^2 \, dx \;\le\; \int_{\frac{1}{2}-\frac{1}{m}}^{\frac{1}{2}+\frac{1}{m}} 1 \, dx \;\le\; \frac{2}{m}$$

Thus, the sequence is L^2-Cauchy. Since the limit is not continuous, the sequence cannot possibly be C^o-Cauchy. Explicitly, $|f_m - f_n|_{C_o} = 1$ for $m \ne n$. ///

[01.7] Show that $C^1[a, b]$ is not complete with the $C^o[a, b]$ metric.

Discussion: One approach is to find a C^o-Cauchy sequence of C^1 functions whose limit is not C^1. For example, in words, a tent function with base $[a, b]$ with vertex at the point $(\frac{a+b}{2}, 1)$ is continuous, but not differentiable. It can be approximate in C^o by tent functions that are smoothed off in tinier-and-tinier intervals around the vertex.

Formulaically, it's a question of writing formulas for (for example) little pieces of pointier-and-pointier parabola pieces to replace the sharp corner at the peak of the tent function.

Losing interest in this approach... Is there a better one? Non-formulaic? Seriously, turning obvious pictures into formulas quickly becomes unrewarding and non-explanatory...

Yes: we should soon prove that $C^\infty[a, b]$ is dense in all the spaces $C^k[a, b]$. This changes the presentation of the question, but annihilates it. ///

[01.8] Show that $C^1[a, b]$ is complete, with the $C^1[a, b]$ metric

$$d(f, g) \;=\; \sup_{a \le x \le b} |f(x) - g(x)| + \sup_{a \le x \le b} |f'(x) - g'(x)|$$

Discussion: For a Cauchy sequence $\{f_i\}$ in $C^k[a, b]$, the pointwise limits $\lim_i f(x)$ and $\lim_i f'(x)$ exist, and are continuous, since the limits are uniform pointwise. The issue is to show that $\lim_i f$ is differentiable, with derivative $\lim_i f'$. That is, for a Cauchy sequence f_n in $C^1[a, b]$, with pointwise limits $f(x) = \lim_n f_n(x)$ and $g(x) = \lim_n f_n'(x)$, we have $g = f'$. By the fundamental theorem of calculus, for any index i,

$$f_i(x) - f_i(a) \;=\; \int_a^x f_i'(t) \, dt$$

Since the f_i' *uniformly* approach g, given $\varepsilon > 0$ there is i_o such that $|f_i'(t) - g(t)| < \varepsilon$ for $i \ge i_o$ and for *all* t in the interval, so for such i

$$\Big| \int_a^x f_i'(t) \, dt - \int_a^x g(t) \, dt \Big| \;\le\; \int_a^x |f_i'(t) - g(t)| \, dt \;\le\; \varepsilon \cdot |x - a| \;\longrightarrow\; 0$$

Thus,

$$\lim_i f_i(x) - f_i(a) \;=\; \lim_i \int_a^x f_i'(t)\,dt \;=\; \int_a^x g(t)\,dt$$

from which $f' = g$. ///

[01.9] Show that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals r_n,

$$C(r) \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le r_n\}$$

is compact if and only if $\sum_n |r_n|^2 < \infty$.

Discussion: Use the *total boundedness* criterion. Given $\varepsilon > 0$, by convergence of $\sum_n \delta_n^2$, there is n_o large enough so that $\sum_{n \ge n_o} \delta_n^2 < \varepsilon^2$. The set

$$C_{n_o} \;=\; \{(z_1, z_2, \ldots, z_{n_o}) \in \mathbb{R}^{n_o} : |z_n| \le \delta_n\}$$

is a compact subset of $\mathbb{R}^{n_o}$, so certainly has a finite cover by open balls of radius ε. Let the centers of these balls be $w_1, \ldots, w_N$. Let $j : \mathbb{R}^{n_o} \to \ell^2$ be the inclusion $j(z_1, \ldots, z_{n_o}) = (z_1, \ldots, z_{n_o}, 0, 0, \ldots)$. Then we claim that the open balls of radius 2ε at $j(w_1), j(w_2), \ldots, j(w_N)$ cover $C(\delta)$. Indeed, given $z = (z_1, z_2, \ldots) \in C(\delta)$, write $z = j(z') + z''$ where $z' = (z_1, \ldots, z_{n_o})$ and $z'' = z - j(z') = (0, \ldots, 0, z_{n_o+1}, \ldots)$. There is at least one of the w_js within ε of z': let w_{j_o} be such. By the triangle inequality for the norm $|\cdot|_{\ell^2}$ on ℓ^2,

$$\begin{aligned} d(z, j(w_{j_o})) \;&=\; |z - j(w_{j_o})|_{\ell^2} \;=\; |j(z') + z'' - j(w_{j_o})|_{\ell^2} \;\le\; |j(z') - j(w_{j_o})|_{\ell^2} + |z''|_{\ell^2} \\ &=\; |z' - w_{j_o}|_{\mathbb{R}^{n_o}} + |z''|_{\ell^2} \;<\; \varepsilon + \varepsilon \end{aligned}$$

Thus, $C(r)$ can be covered by finitely-many open balls of radius 2ε.

Conversely, if $\sum_n r_n^2 = +\infty$, then there are indices $1 \le n_1 < n_2 < \ldots$ such that

$$\sum_{n_k < i \le n_{k+1}} r_n^2 \;\ge\; 1$$

With standard basis $\{e_n\}$, let

$$v_k \;=\; \sum_{n_k < i \le n_{k+1}} r_i \cdot e_i$$

Then for $k \ne \ell$,

$$|v_k - v_\ell|^2 \;=\; \sum_{n_k < i \le n_{k+1}} r_i^2 + \sum_{n_\ell < i \le n_{\ell+1}} r_i^2 \;\ge\; 1 + 1$$

Thus, there are no convergent subsequences, and $C(r)$ is not sequentially compact, so not compact. ///

[01.10] Let $|\cdot|_1$ and $|\cdot|_2$ be two norms on a real or complex vector space X. Suppose that $|x|_1 \ge |x|_2$ for all $x \in X$. Let X_i be the completion of X with respect to the metric associated to $|\cdot|_i$. Show that the identity map $X \to X$ *extends by continuity* to a continuous *injection* $X_1 \to X_2$.

Discussion: As usual, attempt to define the extension-by-continuity S of the identity map by $S(X_1 - \lim x_n) = X_2 - \lim x_n$ for $x_n \in X$. Then we'd want or need to show that it is well-defined, that it is continuous, and linear, *and* that it is injective. All but the injectivity are treated in excruciating detail in the notes.

For injectivity, it is probably best to *not* attempt to prove this directly by purely elementary means. It *is* a significant issue, though, so we'll come back to this later.