(*October 28, 2018*)

Examples discussion 02

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[02.1] The space of continuous functions on $\mathbb{R}$ *going to* 0 *at infinity* is

$$C^o_o(\mathbb{R}) \;=\; \{f \in C^o(\mathbb{R}) : \text{for every } \varepsilon > 0 \text{ there is } T \text{ such that } |f(x)| < \varepsilon \text{ for all } |x| \ge T\}$$

Show that the closure of $C^o_c(R)$ in the space $C^o_{\rm bdd}(\mathbb{R})$ of bounded continuous functions with sup norm, is $C^o_o(\mathbb{R})$.

Discussion: The argument for this is general enough that we can replace $\mathbb{R}$ by a more general topological space X, probably locally compact and Hausdorff so that Urysohn's lemma assures us a good supply of continuous functions for auxiliary purposes. Then $C^o_o(X)$ is defined to be the collection of continuous functions f such that, given $\varepsilon > 0$, there is a compact $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$.

First, show that any $f \in C^o_o(\mathbb{R})$ is a sup-norm limit of functions from $C^o_c(\mathbb{R})$. Given $\varepsilon > 0$, let K be sufficiently large so that $|f(x)| < \varepsilon$ for $x \notin K$. We claim that there is an open $U \supset K$ with compact closure $\overline{U}$ (which would be obvious on $\mathbb{R}$ or $\mathbb{R}^n$). For each $x \in K$, let $U_x \ni x$ be an open set with compact closure (using the local compactness). By compactness of K, there is a finite subcover $K \subset U_{x_1} \cup \ldots \cup U_{x_n}$. Then the closure of $U = U_{x_1} \cup \ldots \cup U_{x_n}$ is compact, as claimed. Then, invoking Urysohn's Lemma, let φ be a continuous function on X taking values in the interval $[0, 1]$, that is 1 on K, and 0 off U, so φ has compact support. Then $\varphi \cdot f$ is continuous and has compact support, and

$$\sup_{x\in X} |f(x) - \varphi(x)\cdot f(x)| \;\le\; \sup_{x\in K} |f(x) - \varphi(x)\cdot f(x)| + \sup_{x\notin K} |f(x) - \varphi(x)\cdot f(x)| \;=\; 0 + \sup_{x\notin K} |f(x) - \varphi(x)\cdot f(x)|$$

$$\le\; \sup |1-\varphi| \cdot \sup_{x\notin K} |f(x)| \;<\; 1\cdot\varepsilon$$

That is, we can approximate f to within ε, as claimed.

On the other hand, now show that any sup-norm Cauchy sequence of $f_n \in C^o_c(X)$ has a pointwise limit f in $C^o_o(X)$. First, on any compact, the limit of the f_n's is *uniform* pointwise, so is continuous on compacts. Since every point $x \in X$ has a neighborhood U_x with compact closure, the pointwise limit is continuous on U_x. Thus, the pointwise limit is continuous at every point, hence continuous. Given $\varepsilon > 0$, take n_o sufficiently large so that $\sup_{x\in X} |f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \ge n_o$. Let K be the support of f_{n_o}. Then

$$\sup_{x\notin K} |f(x)| \;=\; \sup_{x\notin K} |f(x) - f_{n_o}(x)| \;\le\; \sup_{x\in X} |f(x) - f_{n_o}| \;\le\; \varepsilon$$

Thus, the pointwise limit goes to 0 at infinity. ///

[02.2] Show that $|\int_a^b f|^2 \le |b-a| \cdot \int_a^b |f|^2$.

Discussion: This is the Cauchy-Schwarz-Bunyakowsky inequality on $L^2[a, b]$, where the inner product is

$$\langle f, g\rangle \;=\; \int_a^b f\,\overline{g} \;=\; \int_a^b f(x)\,\overline{g(x)}\,dx$$

$$|\int_a^b f|^2 \;=\; \Big|\int_a^b 1\cdot f(x)\,dx\Big|^2 \;\le\; \int_a^b 1 \;\cdot \int_a^b |f|^2 \;=\; |b-a|\cdot \int_a^b |f|^2$$

[02.3] In ℓ^2, show that the unique point in the closed unit ball closest to a point v *not* inside that ball is $v/|v|_{\ell^2}$.

Discussion: The minimum principle assures that there is a *unique* closest point w in the closed unit ball B to v, because B is convex, closed, non-empty, and v is not in B.

First, we show that *any* minimizing point w must be on the boundary of B, that is, $|w| = 1$. Indeed, if not, there is sufficiently small $\varepsilon > 0$ such that the ε-ball around w is inside B. Then we can move from w slightly in the direction of v, getting strictly closer to v than w was: in formulas,

$$\Big|(w+\varepsilon\cdot\frac{v-w}{|v-w|}) - v\Big| \;=\; \Big|(1+\varepsilon\cdot\frac{1}{|v-w|})\cdot(w-v)\Big| \;=\; \Big|(1-\varepsilon\cdot\frac{1}{|w-v|})\cdot(w-v)\Big| \;=\; \Big|(1-\varepsilon\cdot\frac{1}{|w-v|})\Big|\cdot|v-w| \;<\; |v-w|$$

contradiction.

Suppose w (with $|w| = 1$ is closer than $v/|v|$. Then

$$|v|^2 - 2|v| + 1 \;=\; |v - \frac{v}{|v|}|^2 \;>\; |v-w|^2 \;=\; |v|^2 - \langle v,w\rangle - \langle w,v\rangle + |w|^2 \;=\; |v|^2 - \langle v,w\rangle - \langle w,v\rangle + 1$$

Thus,

$$2|v| \;<\; \langle v,w\rangle + \langle w,v\rangle$$

Thus, the sum of the two inner products is *positive*, and by Cauchy-Schwarz-Bunyakowsky:

$$2|v| \;<\; \langle v,w\rangle + \langle w,v\rangle \;=\; |\langle v,w\rangle + \langle w,v\rangle| \;\le\; 2|v|\cdot|w|$$

Thus, $1 < |w|$, which is impossible. ///

[02.4] One form of the *sawtooth* function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\widehat{f}(n)$. From Plancherel-Parseval's theorem for this function, show that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots \;=\; \frac{\pi^2}{6}$$

Discussion: We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0, 2\pi]$. The Fourier coefficients are determined by Fourier's formula

$$\widehat{f}(n) \;=\; \int_0^{2\pi} f(x)\,\frac{e^{-inx}}{\sqrt{2\pi}}\,dx$$

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get

$$\widehat{f}(n) \;=\; \Big[f(x)\cdot\frac{e^{-inx}}{\sqrt{2\pi}\cdot(-in)}\Big]_0^{2\pi} - \int_0^{2\pi} 1\cdot\,\frac{e^{-inx}}{\sqrt{2\pi}\cdot(-in)}\,dx$$

$$= \Big((\pi\cdot\frac{1}{\sqrt{2\pi}\cdot(-in)}) - (-\pi\cdot\frac{1}{\sqrt{2\pi}\cdot(-in)})\Big) - 0 \;=\; \frac{2\pi}{\sqrt{2\pi}\cdot(-in)} \;=\; \frac{\sqrt{2\pi}}{-in}$$

The L^2 norm of f is

$$\int_0^{2\pi}(x-\pi)^2\,dx \;=\; \Big[\frac{(x-\pi)^3}{3}\Big]_0^{2\pi} \;=\; \frac{\pi^3 - (-\pi)^3}{3} \;=\; \frac{2\pi^3}{3}$$

Thus, by Parseval,

$$\sum_{n\neq 0}\Big|\frac{\sqrt{2\pi}}{-in}\Big|^2 \;=\; \frac{2\pi^3}{3}$$

This simplifies first to

$$2\sum_{n\geq 1}\frac{2\pi}{n^2} \;=\; \frac{2\pi^3}{3}$$

and then to

$$\sum_{n\geq 1}\frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

That is, Parseval applied to the sawtooth function evaluates $\zeta(2)$. ///

[02.5] Show that there is *no* $f_o \in C^o[0,1]$ so that, for all $g \in C^o[0,1]$, $\int_0^1 f_o(x)\,g(x)\,dx\rangle = g(\frac{1}{2})$.

Discussion: Here is just one among many possible approaches. By Cauchy-Schwarz-Bunyakowsky in $L^2[0,1]$ with its usual inner product, for every $g \in C^o[0,1]$ we'd have

$$|g(\tfrac{1}{2})| \;=\; \Big|\int_0^1 f_o(x)\,g(x)\,dx\rangle\Big| \;=\; |\langle g, \overline{f}_o\rangle| \leq |g|_{L^2}\cdot|\overline{f}_o|_{L^2}$$

That is, supposedly $g(\frac{1}{2})$ would be bounded by a constant multiple of $|g|_{L^2}$, for every $g \in C^o$. But this is not true: we can make a variety of sequences $\{g_n\}$ of continuous functions with support in $[\frac{1}{2}-\frac{1}{n}, \frac{1}{2}+\frac{1}{n}]$, with $g_n(\frac{1}{2}) = 1$, and with $\sup|g_n| = 1$. Piecewise-linear tent functions of height 1 and base $[\frac{1}{2}-\frac{1}{n}, \frac{1}{2}+\frac{1}{n}]$ would do. The L^2 norms go to 0 as $n \to +\infty$. ///

[02.6] For $c_1 > c_2 > c_3 > \ldots > 0$ a monotone-decreasing sequence of positive reals, with $\lim_n c_n = 0$, show that, for every $0 < x < 2\pi$, $\sum_n c_n\, e^{inx}$ converges.

Discussion: The expression as a Fourier series should not distract us from seeing an instance of the generalized alternating-decreasing criterion again, sometimes called *Dirichlet's criterion*: for a positive real sequence $c_1, c_2, \ldots$ monotone-decreasing to 0, and for a (possibly complex) sequence $b_1, b_2, \ldots$ with *bounded partial sums* $B_n = b_1 + \ldots + b_n$, the sum $\sum_n b_n c_n$ converges. The partial sums $\sum_{n\leq N} e^{2\pi i n x}$ are bounded for $0 < x < 1$, by summing *finite* geometric series:

$$\Big|\sum_{n=-M}^{N} z^n\Big| \;=\; \frac{|z^{-M} - z^{N+1}|}{|1-z|} \;\leq\; \frac{2}{|1-z|}$$

so this criterion applies here.

The proof of the criterion itself is by *summation by parts*, a discrete analogue of integration by parts. That is, rewrite the tails of the sum as

$$\sum_{M\leq n\leq N} b_n c_n \;=\; \sum_{M\leq n\leq N}(B_n - B_{n-1})c_n \;=\; -B_{M-1}c_M + \sum_{M\leq n\leq N} B_n(c_n - c_{n+1}) + B_N c_{N+1}$$

Since the partial sums are bounded, the first and last summand go to 0. Letting β be a bound for all the $|B_n|$, the summation is

$$\Big|\sum_{M\leq n\leq N} B_n(c_n - c_{n+1})\Big| \;\leq\; \sum_{M\leq n\leq N}|B_n|\cdot|c_n - c_{n+1}| \;=\; \sum_{M\leq n\leq N}|B_n|\cdot(c_n - c_{n+1}) \;\leq\; \sum_{M\leq n\leq N}\beta\cdot(c_n - c_{n+1})$$

$$=\; \beta\cdot\sum_{M\leq n\leq N}(c_n - c_{n+1}) \;=\; \beta\cdot(c_M - c_{N+1})$$

by telescoping the series. Again, c_M and c_{N+1} go to 0. ///

[02.7] Let $b = \{b_n\}$ be a sequence of complex numbers, such that there is a bound B such that, for every $c = \{c_n\} \in \ell^2$, $|\sum_n b_n c_n| \le B \cdot |c|_{\ell^2}$. Show that $b \in \ell^2$.

Discussion: The assumed inequality says that $\lambda(c) = \sum_n b_n c_n$ is a *bounded* linear functional on ℓ^2. By Riesz-Fréchet, there is $a = (a_1, a_2, \ldots) \in \ell^2$ such that $\lambda(c) = \sum_n a_n c_n$ for all $c \in \ell^2$. Then, with $\{e_n\}$ the standard (Hilbert-space) basis for ℓ^2, $b_n = \lambda(e_n) = a_n$ proves that $a = b$, so $b \in \ell^2$. ///

[02.8] For a vector subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the topological closure of W.

Discussion: Let $\lambda_x(v) = \langle v, x\rangle$ for $x, v \in V$. Then $W^\perp = \bigcap_{w\in W} \ker \lambda_w$. Similarly, $(W^\perp)^\perp = \bigcap_{x \in W^\perp} \ker \lambda_x$. From the discussion in the Riesz-Fréchet theorem, or directly via Cauchy-Schwarz-Bunyakowsky, each λ_x is continuous, so $\ker \lambda_x = \lambda_x^{-1}(\{0\})$ is closed, since $\{0\}$ is closed. (One might check that the kernel of a linear map is a vector subspace.) An arbitrary intersection of closed sets is closed, so $(W^\perp)^\perp$ is closed.

Certainly $(W^\perp)^\perp \supset W$, because for each $w \in W$, $\langle x, w\rangle = 0$ for all $x \in W^\perp$. Thus, $(W^\perp)^\perp$ is a closed subspace, containing W. Being a closed subspace of a Hilbert space, $(W^\perp)^\perp$ is a Hilbert space itself. If $(W^\perp)^\perp$ were strictly larger than the topological closure $\overline{W}$ of W, then there would be $0 \ne y \in (W^\perp)^\perp$ orthogonal to $\overline{W}$. Then y would be orthogonal to W itself, so $0 \ne y \in W^\perp$, contradicting $0 \ne y \in (W^\perp)^\perp$. ///

[02.9] Find two *dense* vector subspaces X, Y of ℓ^2 such that $X \cap Y = \{0\}$. (And, if you need further entertainment, can you find countably-many dense vector subspaces X_n such that $X_m \cap X_n = \{0\}$ for $m \ne n$?)

Discussion: First, as a variant that refers to more natural constructions, but requires non-trivial proofs to fully validate it, we can make two dense subspaces of $L^2[0,1]$ which intersect just at $\{0\}$. Namely, the vector space of all finite Fourier series, and the vector space of all polynomials (restricted to $[0,1]$). We need to know that the appropriate exponentials (or sines and cosines) give a Hilbert space basis of $L^2[0,1]$, and also Weierstraß' result on the density of polynomials in $C^o[0,1]$, hence (depending on our definitional set-up) in $L^2[0,1]$.

A more elementary, but trickier, approach is the following. Let X be the vector space of *finite* linear combinations of the standard Hilbert space basis $\{e_n\}$. This is a natural subspace. For the other subspace Y, *some* sort of trickery seems to be needed, either in specification of Y itself so as to make verification of $X \cap Y = \{0\}$ easy, or a simpler specification of Y but with complicated verification that $X \cap Y = \{0\}$, or both.

One possibility involves Sun-Ze's theorem (sometimes called the Chinese Remainder Theorem), namely, that for a finite collection of mutually relatively prime integers $N_1, \ldots, N_k$, and for integers $b_1, \ldots, b_k$ there exists $x \in \mathbb{Z}$ such that $x = b_k \bmod N_k$. Further, this x can be arbitrarily large, by adding multiples of the product $N_1 ... N_k$ to it. Let p_n be the n^{th} prime number, and put

$$v_n \;=\; e_n + \sum_{k\ge 1} \frac{1}{kp_n} \cdot e_{kp_n}$$

Of course, we claim that no (non-zero) finite linear combination $y = \sum_n c_n \cdot v_n$ is in X. That is, we claim that for any such non-zero linear combination, there are arbitrarily large indices ℓ such that $\langle y, e_\ell\rangle \ne 0$. Let n_o be the largest index n such that $c_n \ne 0$. Invoking Sun-Ze's theorem, there exist $\ell \ge n_o$ such that $\ell = 1 \bmod p_i$ for $i < n_o$ and $\ell = 0 \bmod p_{n_o}$. Then

$$\langle y, e_\ell\rangle \;=\; \sum_n \Big(\frac{1}{n}\langle e_n, e_\ell\rangle + \sum_k \frac{1}{kp_n}\langle e_{kp_n}, e_\ell\rangle\Big) \;=\; \sum_{n<n_o} 0 \;+\; \frac{1}{\ell}\;\textit{not}\; = 0$$

This proves that $X \cap Y = \{0\}$.

Certainly X is dense, because every vector in ℓ^2 is an infinite sum of vectors from X, that is, an ℓ^2 limit of finite linear combinations of vectors from X.

To see that Y is dense, observe that applying an *infinite* version of Gram-Schmidt to the vectors v_n produces the standard basis e_n. That is, the e_n's are *infinite* linear combinations of the v_n's, so Y is dense. (Yes, there is an issue about *convergence* in an infinite version of Gram-Schmidt, in general!) ///
