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Examples discussion 03

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-disc-03.pdf]

[03.1] Show that the characteristic function of a measurable set is a measurable function.

Discussion: For non-empty open $U \subset \mathbb{R}$, $\chi_E^{-1}(U)$ is the measurable set ϕ if U does not contain either 0 or 1. If $U \ni 1$ but $U \not\ni 0$, then $\chi_E^{-1}(U) = E$, which is measurable. If $U \ni 0$ but $U \not\ni 1$, then $\chi_E^{-1}(U) = E^c$, the complement of E, which is measurable. If U contains both 0 and 1, then $\chi_E^{-1}(U)$ is the whole domain space, which is measurable. ///

[03.2] For measurable $E \subset [0, 2\pi]$, show that $\lim_n \int_E e^{-inx}\,dx = 0$ as $n \to \infty$ ranging over integers.

Discussion: This is a relatively easy instance of a *Riemann-Lebesgue lemma*, namely, that Fourier coefficients of an L^2 function on $[0, 2\pi]$ go to 0. Here, the L^2 function is the characteristic function of E.

In fact, this relatively easy Riemann-Lebesgue lemma does not even need the completeness of exponentials in L^2, but only Bessel's inequality. ///

[03.3] For $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, show that there is a constant C (depending on f) such that

$$\Big| \int_{t-\delta}^{t+\delta} f(x)\,dx \Big| \;<\; C \cdot \sqrt{\delta}$$

Discussion: Let h_δ be the characteristic function of $[t-\delta, t+\delta]$. By Cauchy-Schwarz-Bunyakowsky

$$\Big| \int_{t-\delta}^{t+\delta} f \Big| \;=\; |\langle f, h_\delta\rangle_{L^2}| \;\le\; |f|_{L^2} \cdot |h_\delta|_{L^2} \;=\; |f|_{L^2} \cdot \sqrt{2\delta}$$

The case of conjugate exponents $\frac{1}{p} + \frac{1}{q} = 1$ is the same, using Hölder's inequality rather than Cauchy-Schwarz-Bunyakowsky. There is no immediate analogue for L^1, although a weaker result is possible, as in the next example. ///

[03.4] For $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, show that, given $\varepsilon > 0$, there $\delta > 0$ such that

$$\Big| \int_{t-\delta}^{t+\delta} f(x)\,dx \Big| \;<\; \varepsilon$$

Discussion: Let $S_n = \{x : \frac{1}{n+1} \le |x - t| < \frac{1}{n}\}$. Then

$$\Big| \sum_{n \ge 1} \int_{S_n} f \Big| \;\le\; \sum_{n\ge 1} \int_{S_n} |f| \;\le\; |f|_{L^1}$$

Thus, the sum of non-negative terms $\sum_{n\ge 1} \int_{S_n} |f|$ is convergent, so the tails $\sum_{n \ge N} \int_{S_n} |f|$ go to 0 as $N \to +\infty$. Thus,

$$\Big| \int_{|x-t|\le 1/N} f \Big| \;\le\; \int_{|x-t|\le 1/N} |f| \;=\; \sum_{n\ge N} \int_{S_n} |f|$$

goes to 0 as $N \to +\infty$. Then this idea can be applied to $\int_{|x-t|<\delta} |f|^p$ in the previous example. ///

[03.5] For non-negative real-valued f, show that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x)\, dx$$

(whether or not the integrals are finite).

Discussion: Among other possibilities, this is an instance of application of Lebesgue's Monotone Convergence Theorem, since $f(x)\, e^{-\varepsilon x^2} \le f(x)\, e^{-\varepsilon' x^2}$ for $\varepsilon < \varepsilon'$. Thus,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x) \lim_{\varepsilon} e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x)\cdot 1\, dx$$

as claimed. ///

[03.6] For $f \in L^1(\mathbb{R})$, show that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x)\, dx$$

Discussion: Among other possibilities, this is an instance of application of Lebesgue's Dominated Convergence Theorem, since $|f(x)\, e^{-\varepsilon x^2}| \le |f(x)| \in L^1(\mathbb{R})$. Thus,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x) \lim_{\varepsilon} e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x)\cdot 1\, dx$$

as claimed. ///

[03.7] (*Comparing L^p spaces*) Let $1 \le p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

Discussion: Take $p < p'$. We claim that $L^p[a,b] \supset L^{p'}[a,b]$, with proper containment. The function f that is $(x-a)^{-\frac{1}{p'}}$ on $(a,b]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p. Given $f \in L^{p'}[a,b]$, let E be the set of $x \in [a,b]$ where $|f(x)| \ge 1$. Then $\int_a^b |f|^{p'} < \infty$ if and only if $\int_E |f|^{p'} < \infty$. On E, $|f|^p < |f|^{p'}$, so $\int_E |f|^p < \infty$, and then also $\int_a^b |f|^p < \infty$, so $f \in L^p[a,b]$. ///

We claim that $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$ are not comparable for $p \ne p'$. Take $1 \le p < p'$. On one hand, $1/(1+|x|)^{1/p'+\varepsilon}$ is in $L^{p'}$ for all $\varepsilon > 0$, but not in L^p for ε small enough so that $\frac{1}{p'} + \varepsilon < \frac{1}{p}$. On the other hand, the function f that is $x^{-\frac{1}{p'}}$ on $(0,1]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p.

We claim that for $1 \le p < p' < \infty$, $\ell^p \subset \ell^{p'}$, with strict containment. Indeed, $f(n) = 1/n^p$ is not in ℓ^p, but is in $\ell^{p'}$. Let $E = \{n \in \{1,2,\ldots\} : |f(n)| < 1\}$. Then $f \in \ell^p$ if and only if the *complement* of E is finite, and if $\sum_{n\in E} |f(n)|^p < \infty$. Certainly $|f(n)|^p > |f(n)|^{p'}$ for $n \in E$, and the complement of E is finite, so $\sum_{n\in E} |f(n)|^{p'} < \sum_{n\in E} |f(n)|^p$, and $f \in \ell^{p'}$. ///

[03.8] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\le\; w_1 a_1 + \ldots + w_n a_n$$

Discussion: This is a corollary of Jensen's inequality, similar to the arithmetic-geometric mean, but with unequal weights. Namely, let $X = \{1, 2, \ldots, n\}$ with measure $\mu(i) = w_i$, and function $f(i) = \log a_i$. Then Jensen's inequality is

$$\exp\Big(\sum_{i=1}^n w_i \cdot \log a_i\Big) \;\le\; \sum_{i=1}^n w_i \cdot e^{\log a_i}$$

which simplifies to the assertion. ///

[03.9] (*Collecting Fourier transform pairs*) Compute the Fourier transforms of

$$\chi_{[a,b]} \qquad\qquad e^{-\pi x^2} \qquad\qquad f(x) = \begin{cases} e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x \le 0) \end{cases}$$

Discussion: The first of these is direct:

$$\widehat{\chi_{[a,b]}}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x} \chi_{[a,b]}(x)\, dx \;=\; \int_a^b e^{-2\pi i\xi x}\, dx \;=\; \begin{cases} \dfrac{e^{-2\pi i\xi b} - e^{2\pi i\xi a}}{-2\pi i\xi} & (\text{for } \xi \ne 0) \\ b-a & (\text{for } \xi = 0) \end{cases}$$

Since the latter function is *not* in $L^1(\mathbb{R})$, but *is* in $L^2(\mathbb{R})$, we define its Fourier transform (or inverse Fourier transform) *indirectly*, via either the inversion theorem, or by extending-by-continuity via Plancherel, expressing the function as an L^2 limite of L^1 functions.

The third is similarly direct:

$$\widehat{f}(\xi) \;=\; \int_0^\infty e^{-2\pi i\xi x}\, e^{-x}\, dx \;=\; \int_0^\infty e^{-(2\pi i\xi+1)x}\, dx \;=\; \Big[\frac{e^{-(2\pi i\xi+1)x}}{-(2\pi i\xi+1)}\Big]_0^\infty \;=\; \frac{1}{2\pi i\xi+1}$$

Again, the latter function is not in L^1, but is in L^2, so its Fourier transform is most conveniently defined indirectly.

The Gaussian's Fourier transform is less trivial to evaluate, but is a very important example to have in hand, with many different applications throughout mathematics. One approach is as follows. Letting $f(x) = e^{-\pi x^2}$,

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\, e^{-\pi x^2}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi(x^2+2i\xi x)}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2-\pi\xi^2}\, dx \;=\; e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx$$

by completing the square. The unobvious claim is that the integral does not depend on ξ, and, in fact, has value 1. Perhaps the optimal approach here is to observe that the integral is equal to a complex contour integral:

$$\int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2}\, dx \;=\; \int_{i\xi-\infty}^{i\xi+\infty} e^{-\pi z^2}\, dz$$

along the line $\mathrm{Im}(z) = i\xi$. Given the good decay of the integrand as $|\mathrm{Re}(z)| \to \infty$, by Cauchy-Goursat theory, the contour can be *moved* to integration along the real line, giving

$$\int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2}\, dx \;=\; \int_{i\xi-\infty}^{i\xi+\infty} e^{-\pi z^2}\, dz \;=\; \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx \;=\; 1$$

The fact that the latter integral has value 1 comes from the usual trick involving polar coordinates:

$$\Big(\int_{-\infty}^\infty e^{-\pi x^2}\, dx\Big)^2 \;=\; \int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\, dx\, dy \;=\; \int_0^{2\pi}\int_0^\infty e^{-\pi r^2}\, r\, dr\, d\theta \;=\; 2\pi \int_0^\infty e^{-\pi r^2}\, r\, dr$$

Replacing r by $\sqrt{t}$, this is

$$\pi \int_0^\infty e^{\pi t}\, dt \;=\; \pi \cdot \frac{1}{\pi} \;=\; 1$$

Thus, with the present normalization of Fourier transform and corresponding normalization of Gaussian, the Gaussian is its own Fourier transform. ///

[03.10] Compute $\int_{\mathbb{R}} \Big(\frac{\sin x}{x}\Big)^2 \, dx$. (*Hint:* do not attempt to do this directly, nor by complex analysis.)

Discussion: From a standard stock of easy Fourier transforms, the Fourier transform of a characteristic function of a symmetrical interval is very close to the given function:

$$\widehat{\mathrm{ch}_{[-1,1]}}(\xi) \;=\; \int_{-1}^{1} e^{-2\pi i\xi x} \, dx \;=\; \frac{e^{-2\pi i\xi} - e^{2\pi i\xi}}{-2\pi i\xi} \;=\; \frac{\sin 2\pi\xi}{\pi\xi}$$

Applying Plancherel, we have

$$2 \;=\; \int_{\mathbb{R}} |\mathrm{ch}_{[-1,1]}|^2 \;=\; \int_{\mathbb{R}} \Big(\frac{\sin 2\pi\xi}{\pi\xi}\Big)^2 \, d\xi$$

The change of variables replacing ξ by $\xi/2\pi$ gives

$$2 \;=\; \int_{\mathbb{R}} \Big(\frac{\sin\xi}{\xi/2}\Big)^2 \, \frac{d\xi}{2\pi} \;=\; \frac{2}{\pi} \int_{\mathbb{R}} \Big(\frac{\sin\xi}{\xi}\Big)^2 \, d\xi$$

Thus, the desired integral is π. ///

[03.11] Let $E \subset \mathbb{R}$ be a measurable set with finite measure. Show that $\int_E \cos(tx) \, dx \to 0$ as $t \to +\infty$.

Discussion: This is an instance of the more substantial Riemann-Lebesgue Lemma for functions in $L^1(\mathbb{R})$. The characteristic/indicator function χ_E of E is in $L^1(\mathbb{R})$, since the measure of E is finite. Thus, the Fourier transform of χ_E is continuous and goes to 0 at infinity (by Riemann-Lebesgue). This immediately gives the corresponding vanishing for sines and cosines. ///