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Examples discussion04

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[04.1] With $g(x) = f(x + x_o)$, express $\widehat{g}$ in terms of $\widehat{f}$, for $f \in L^1(\mathbb{R}^n)$.

Discussion: For $f \in \mathscr{S}(\mathbb{R}^n)$, the literal integral computes the Fourier transform:

$$\widehat{g}(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\, g(x)\, dxn \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\, f(x + x_o)\, dx$$

Replacing x by $x - x_o$ in the integral gives

$$\widehat{g}(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot(x - x_o)}\, f(x)\, dx \;=\; e^{2\pi i\xi\cdot x_o} \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\, f(x)\, dx \;=\; e^{2\pi i\xi\cdot x_o} \cdot \widehat{f}(\xi)$$

The precise corresponding statement for tempered distributions cannot refer to pointwise values. Write ψ_{x_o} for the function $\xi \to e^{2\pi i\xi\cdot x_o}$. Since ψ_{x_o} is bounded, for a tempered distribution u, $\psi_{x_o} \cdot u$ is the tempered distribution described by

$$(\psi_{x_o} \cdot u)(\varphi) \;=\; u(\psi_{x_o}\, \varphi) \qquad\qquad (\text{for } \varphi \in \mathscr{S})$$

This is compatible with multiplication of (integrate-against-) functions $\mathscr{S} \subset \mathscr{S}^*$. Also, let translation $u \to T_{x_o}u$ be defined by $(T_{x_o}u)(\varphi) = u(T_{-x_o}\varphi)$, again compatibly with integration against Schwartz functions. In these terms, the above argument shows that

$$(T_{x_o}f)\widehat{\ } \;=\; \psi_{x_o} \cdot \widehat{f} \qquad\qquad (\text{for } f \in \mathscr{S})$$

This formulation avoids reference to pointwise values, and thus could make sense for tempered distributions.

One argument is *extension by continuity*: Fourier transform is a continuous map $\mathscr{S}^* \to \mathscr{S}^*$, as is translation $u \to T_{x_o}u$, so the identity extends by continuity to all tempered distributions. ///

Another argument is by *duality*: first,

$$(T_{x_o}u)\widehat{\ }(\varphi) \;=\; (T_{x_o}u)(\widehat{\varphi}) \;=\; u(T_{-x_o}\,\widehat{\varphi}) \;=\; u\Big((\psi_{x_o} \cdot \varphi)\widehat{\ }\Big)$$

by applying the identity to $\varphi, \widehat{\varphi} \in \mathscr{S}$. Going back, this is

$$\widehat{u}(\psi_{x_o} \cdot \varphi) \;=\; (\psi_{x_o} \cdot \widehat{u})(\varphi) \qquad\qquad (\text{for all } \varphi \in \mathscr{S})$$

Altogether, $(T_{x_o}u)\widehat{\ } = \psi_{x_o} \cdot \widehat{u}$. ///

[04.2] Let $\{b_n\}$ be a sequence of complex numbers. Suppose that $\sum_n a_n b_n$ converges for every $\{a_n\} \in \ell^2$. Show that $\{b_n\} \in \ell^2$.

Discussion: This is an example application of *uniform boundedness*, also known as the *Banach-Steinhaus* theorem. Namely, here, for a collection $\{\lambda_n\}$ of continuous linear functionals on ℓ^2, *either* the linear-functional norms $|\lambda_n| = \sup_{|v|\le 1} |\lambda_n(v)|$ are uniformly bounded, *or* there is $v \in \ell^2$ such that $|\lambda_n(v)| \to +\infty$.

Here, let $\lambda_n(\{a_n\}) = \sum_{i\le n} a_n b_n$. Since the corresponding infinite sum converges, by assumption, the absolute values do *not* go to infinity, so the $|\lambda_n|$'s are *uniformly* bounded, by some $C < +\infty$. Let

$$v(n) \;=\; (\overline{b}_1,\, \overline{b}_2,\, \ldots,\, \overline{b}_{n-1},\, \overline{b_n},\, 0,\, 0,\, \ldots)$$

and note that $w(n) = v(n)/|v(n)|$ is in the closed unit ball in ℓ^2. Thus, for every n,

$$\sum_{i\le n} \frac{\bar{b}_i}{\sqrt{\sum_{j\le n}|b_j|^2}} \cdot b_i \;\le\; C$$

Simplifying, this gives $\sum_{i\le n} |b_i|^2 \le C^2$ for all n. Thus, $\sum_i |b_i|^2$ converges, so $\{b_i\} \in \ell^2$. ///

[04.3] Let g be a measurable $[0, +\infty]$-value function on $[a, b]$ such that, for every $f \in L^2[a, b]$, $\int_a^b |f(x)\, g(x)|\, dx < \infty$. Show that $g \in L^2[a, b]$.

Discussion: This is another instance of application of uniform boundedness.

For g almost-everywhere 0, we're done. So we can suppose that g is *not* almost-everywhere 0. Let

$$g_n(x) \;=\; \begin{cases} g(x) & (\text{for } |g(x)| \le n) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

We apply uniform boundedness to the functionals $\lambda_n(f) = \int_a^b g_n \cdot f$, to conclude that there is a *uniform* bound C such that $|\lambda_n(f)| \le C$ for every f in $L^2[a, b]$ with $|f|_{L^2} \le 1$.

Since g is not almost-everywhere 0, for sufficiently large n the functions g_n are not almost-everywhere 0. Thus, for large-enough n, we can let

$$h_n(x) \;=\; \frac{g_n(x)}{\sqrt{\int_a^b |g_n|^2}}$$

(with non-zero denominator). Then

$$\Big|\int_a^b g(x) \cdot h_n(x)\, dx\Big| \;\le\; C$$

Since $g(x) \cdot g_n(x) = 0$ when $g(x) > n$, this gives

$$\frac{\int_a^b g_n(x)^2\, dx}{\sqrt{\int_a^b g_n(x)^2\, dx}} \;\le\; C$$

From this, $\int_a^b |g_n(x)|^2\, dx \le C^2$ for all n, so $g \in L^2$. ///

[04.4] Give a *persuasive* proof that the function

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this to make a *smooth step function*: 0 for $x \le 0$ and 1 for $x \ge 1$, and goes monotonically from 0 to 1 in the interval $[0, 1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1, 2, 3, \ldots\}$: for each n, $f_n(x) = 1$ for $x \in [-n, n]$, $f_n(x) = 0$ for $x \notin [-(n + 1), n + 1]$, and f_n goes monotonically from 0 to 1 in $[-(n + 1), -n]$ and monotonically from 1 to 0 in $[n, n + 1]$.

Discussion: In $x > 0$, by induction, the derivatives are finite linear linear combinations of functions of the form $x^{-n}e^{-1/x}$. It suffices to show that $\lim_{x\to 0^+} x^{-n}e^{-1/x} = 0$. Equivalently, that $\lim_{x\to+\infty} x^n e^{-x} = 0$, which follows from $e^{-x} = 1/e^x$, and

$$x^{-n}e^{-1/x} \;=\; \frac{x^n}{e^x} \;=\; \frac{x^n}{\sum_{m\ge 0}\frac{x^m}{m!}} \;\le\; \frac{x^n}{\frac{x^{n+1}}{(n+1)!}} \;\longrightarrow\; 0 \qquad (\text{as } x \to +\infty)$$

(This is perhaps a little better than appeals to L'Hospital's Rule.) Thus, f is smooth at 0, with all derivatives 0 there. ///

Next, we make a *smooth bump function* by

$$b(x) \;=\; \begin{cases} 0 & (\text{for } x \le -1) \\ e^{\frac{1}{x^2-1}} & (\text{for } -1 < x < 1) \\ 0 & (\text{for } x \ge 1) \end{cases}$$

A similar argument to the previous shows that this is smooth. Renormalize it to have integral 1 by

$$\beta(x) \;=\; \frac{b(x)}{\int_{-1}^{1} b(t)\, dt}$$

Then $\int_{-1}^{x} \beta(t)\; dt$ is a smooth (monotone) step function that goes from 0 at -1 to 1 at 1. The minor modification $s(x) = 2\int_{-1}^{x} \beta(2t-1)\; dt$ gives a smooth (monotone) step function going from 0 at 0 to 1 at 1. ///

Then $s(x+n+1)$ is a smooth, monotone step function going up from 0 to 1 in $[-n-1,-n]$, and $s(n+1-x)$ for $n \in \mathbb{Z}$ is a smooth, monotone step function going *down* from 1 to 0 in $[n, n+1]$. Thus, the product $f_n(x) = s(x+n+1)\cdot s(n+1-x)$ is the desired smooth cut-off function. ///

[04.5] Give an explicit non-zero function f such that $\int_{\mathbb{R}} x^n\, f(x)\; dx = 0$, for all $n = 0, 1, 2, \ldots$.

Discussion: We choose to find a *Schwartz* function f meeting the condition, since success in finding such f in such a relatively small class of nice functions will be a stronger result than find such f in a larger class of less-nice functions.

For Schwartz (and other) functions g, $\int_{\mathbb{R}} g(x)\, dx = \widehat{g}(0)$. Thus, the requirement on f is that

$$0 \;=\; (\widehat{x^n f})(0) \;=\; (-2\pi i)^{-n} \Big(\frac{d}{dx}\Big)^n \widehat{f}(0)$$

Thus, the requirement on $f \in \mathscr{S}$ is equivalent to the vanishing of all derivatives of $\widehat{f}$ at 0. Taking $\widehat{f}$ to be a smooth bump function with support not including 0 would suffice, for example,

$$\widehat{f}(x) \;=\; \begin{cases} e^{1/(x-1)(x-3)} & (\text{for } 1 < x < 3) \\ 0 & (\text{otherwise}) \end{cases}$$

and then f is the inverse Fourier transform of $\widehat{f}$:

$$f(x) \;=\; \int_1^3 e^{2\pi i \xi x} e^{1/(\xi-1)(\xi-3)}\; d\xi$$

Note that f cannot be compactly supported and meet the requirement, because in that case $\widehat{f}$ is an entire (holomorphic) function (in the Paley-Wiener space), which cannot vanish to infinite order at any point (without being identically 0). ///

[04.6] Show that $\chi_{[a,b]} * \chi_{[c,d]}$ is a piecewise-linear function, and express it explicitly.

Discussion: Once enunciated, this fact (and the explicit expression) should be just a matter of book-keeping. We do assume that $a \le b$ and $c \le d$. Also, by symmetry, without loss of generality we can suppose that $|b-a| \ge |d-c|$. This is used in the treatment of cases below.

$$(\chi_{[a,b]} * \chi_{[c,d]})(x) \;=\; \int_{\mathbb{R}} \chi_{[a,b]}(x-y)\cdot\chi_{[c,d]}(y)\,dy \;=\; \int_c^d \chi_{[a,b]}(x-y)\,dy$$

$$= \int_c^d \chi_{[a-x,b-x]}(-y)\,dy \;=\; \int_{-d}^{-c} \chi_{[a-x,b-x]}(y)\,dy \;=\; \text{meas}\Big([-d,-c]\cap[a-x,b-x]\Big)$$

Looking at the cases of overlap, using $b-a \ge d-c$, this is

$$\begin{cases} 0 & (\text{for } b-x \le -d, \text{ that is, } [a-x,b-x] \text{ is to the left of } [-d,-c]) \\ (b-x)-(-d) & (\text{for } a-x \le -d \le b-x \le -c) \\ (-c)-(-d) & (\text{for } a-x \le -d \le -c \le b-x, \text{ that is, } [-d,-c] \subset [a-x,b-x]) \\ (-c)-(a-x) & (\text{for } -d \le a-x \le -c \le b-x) \\ 0 & (\text{for } a-x \ge -c, \text{ that is, } [a-x,b-x] \text{ is to the right of } [-d,-c]) \end{cases}$$

$$= \begin{cases} 0 & (\text{for } x \ge b+d) \\ b+d-x & (\text{for } \max(a+d,b+c) \le x \le b+d) \\ d-c & (\text{for } a+d \le x \le b+c) \\ -a-c+x & (\text{for } a+c \le x \le \min(b+c,a+d)) \\ 0 & (\text{for } x \le a+c) \end{cases}$$

We used the fact that $b-a \ge d-c$ implies $a-c \le b-d$. It is useful to consider the special configuration $[a,b]=[-A,A]$ and $[c,d]=[-B,B]$ with $A \ge B \ge 0$: the convolution is

$$\begin{cases} 0 & (\text{for } x \ge A+B) \\ A+B-x & (\text{for } A-B \le x \le A+B) \\ 2B & (\text{for } -A+B \le x \le A-B) \\ A+B+x & (\text{for } -A-B \le x \le -A+B) \\ 0 & (\text{for } x \le -A-B) \end{cases}$$

In particular, the convolution is supported inside $[-A-B, A+B]$. Similarly, for f and g supported in $[-a,a]$ and $[-b,b]$, the convolution is supported in $[-a-b,\, a+b]$. ///

[04.7] Compute $e^{-\pi x^2} * e^{-\pi x^2}$ and $\dfrac{\sin x}{x} * \dfrac{\sin x}{x}$. (Be careful what you say: $\frac{\sin x}{x}$ is not in $L^1(\mathbb{R})$, so there are potential problems with convolution.)

Discussion: The idea is to invoke $f*g = (\widehat{f}\cdot\widehat{g})^{\widehat{}}$ for *even* functions $f,g \in L^1$, since for even functions the inverse Fourier transform is the same as the forward Fourier transform. Conveniently, Gaussians are in

$L^1 \cap L^2$, and, from above, have Fourier transforms which are again Gaussians:

$$\widehat{e^{-\pi a x^2}}(\xi) \;=\; \frac{1}{\sqrt{a}} e^{-\pi \xi^2/a} \qquad\qquad \text{(for } a > 0\text{)}$$

so

$$e^{-\pi x^2} * e^{-\pi x^2}(\xi) \;=\; \widehat{e^{-\pi x^2} \cdot e^{-\pi x^2}}(\xi) \;=\; \widehat{e^{-2\pi x^2}}(\xi) \;=\; \frac{1}{\sqrt{2}} e^{-\pi\xi^2/2}$$

For the other example, the bound $|f * g|_{L^1} \le |f|_{L^p} \cdot |g|_{L^q}$ for conjugate exponents p, q shows that $f * g \in L^1$ for $f, g \in L^2$. Thus, the same identity holds for $f, g \in L^2$, with the Plancherel extension of Fourier transform. That is, $\widehat{f}$ and $\widehat{g}$ need not be the literal integrals for the Fourier transform, but its extension by continuity to L^2. Above, we computed the Fourier transform of characteristic functions of intervals:

$$\widehat{\chi_{[-a,a]}}a(\xi) \;=\; \frac{\sin 2\pi a \xi}{\pi \xi}$$

Thus,

$$(\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]})\widehat{\;}\,(\xi) \;=\; \frac{\sin \xi}{\xi}$$

Then

$$\begin{aligned}\Big(\frac{\sin x}{x} * \frac{\sin x}{x}\Big)(\xi) \;&=\; \Big((\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]}) \cdot (\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]})\Big)\widehat{\;}(\xi) \\ &=\; \pi \cdot (\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]})\widehat{\;}(\xi) \;=\; \pi \cdot \frac{\sin \xi}{\xi}\end{aligned}$$

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[04.8] Evaluate the *Borwein integral*

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\, dx$$

Discussion: View this as an inner product and invoke Plancherel:

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\, dx \;=\; \Big\langle \frac{\sin x}{x},\, \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \Big\rangle \;=\; \Big\langle \Big(\frac{\sin x}{x}\Big)\widehat{\;},\, \Big(\frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\Big)\widehat{\;} \Big\rangle$$

Since Fourier transform converts pointwise multiplication to convolution, this is

$$\Big\langle \Big(\frac{\sin x}{x}\Big)\widehat{\;},\, \Big(\frac{\sin x/3}{x/3}\Big)\widehat{\;} * \Big(\frac{\sin x/5}{x/5}\Big)\widehat{\;} \Big\rangle$$

We have computed that

$$\widehat{\chi_{[-a,a]}}(\xi) \;=\; \frac{\sin 2\pi a\xi}{\pi\xi} \;=\; 2a \cdot \frac{\sin 2\pi a \xi}{2\pi a \xi}$$

That is, by linearity of Fourier transform,

$$\Big(\frac{1}{2a}\chi_{[-a,a]}\Big)\widehat{\;}\,(\xi) \;=\; \frac{\sin(2\pi a)\xi}{(2\pi a)\xi}$$

By Fourier inversion, noting that $\frac{\sin x}{x}$ is not in L^1, only in L^2, so the inverse transform is not necessarily the literal integral,

$$\Big(\frac{\sin(2\pi a)\xi}{(2\pi a)\xi}\Big)\widehat{\;}\,(x) \;=\; \frac{1}{2a}\chi_{[-a,a]}(x)$$

Replacing a by $a/2\pi$ gives

$$\Big(\frac{\sin a\xi}{a\xi}\Big)^{\widehat{}}(x) \;=\; \frac{\pi}{a}\,\chi_{[-\frac{a}{2\pi},\,\frac{a}{2\pi}]}(x)$$

We will use $a = 1, \frac{1}{3}$, and $\frac{1}{5}$. The relevant convolution was also computed above, but all we need is the fact that the support of

$$3\pi\,\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * 5\pi\,\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}$$

is inside the interval $[-\frac{1}{6\pi} - \frac{1}{10\pi},\; \frac{1}{6\pi} + \frac{1}{10\pi}]$. Thus, the integral of three *sinc* functions is equal to

$$\int_{\mathbb{R}} \pi\chi_{[\frac{-1}{2\pi},\frac{1}{2\pi}]}(x)\cdot\Big(3\pi\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * 5\pi\,\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\,dx \;=\; \pi\cdot 3\pi\cdot 5\pi\int_{-1/\pi}^{1/\pi}\Big(\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * \chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\,dx$$

$$=\; \pi\cdot 3\pi\cdot 5\pi\int_{\mathbb{R}}\Big(\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * \chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\,dx$$

since $[-1/2\pi,\; 1/2\pi]$ contains the support of the convolution. Observing that (invoking Fubini-Tonelli as necessary),

$$\int_{\mathbb{R}}(f*g)(x)\,dx \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)\,dx\,dy \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\,dx\,dy \;=\; \int_{\mathbb{R}} f(x)\,dx\cdot\int_{\mathbb{R}} g(x)\,dy$$

the integral of the convolution is

$$\int_{\mathbb{R}}\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]}\cdot\int_{\mathbb{R}}\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]} \;=\; \frac{1}{3\pi}\cdot\frac{1}{5\pi}$$

Thus, the whole is

$$\pi\cdot 3\pi\cdot 5\pi\cdot\frac{1}{3\pi}\cdot\frac{1}{5\pi} \;=\; \pi$$

Similarly, the integral of $f_1 * \ldots f_n$ is the product of the integrals $\int f_i$. With the support of f_i inside $[-a_i, a_i]$, the support of the convolution is inside $[-a_1 - \ldots - a_n,\; a_1 + \ldots + a_n]$. Thus, since $\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{13} < 1$, the same argument shows that

$$\int_{\mathbb{R}}\frac{\sin x}{x}\cdot\frac{\sin x/3}{x/3}\cdot\frac{\sin x/5}{x/2n+1}\,dx \;=\; \pi \qquad\qquad (\text{for } 2n+1 = 3,5,7,9,11,13)$$

but for $2n+1 = 15$, the support of the Fourier transform of $\frac{\sin x}{x}$ no longer contains the support of the convolution. ///

[04.9] For $f \in \mathscr{S}$, show that

$$\lim_{\varepsilon\to 0^+} f(x) * \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} \;=\; f(x)$$

Discussion: It suffices to show that the functions $\varphi_\varepsilon(x) = e^{-\pi x^2/\varepsilon}$ form an approximate identity, in a not-quite-strictest sense that their masses bunch up at 0, although their supports to do not shrink to $\{0\}$.

We know that $\int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1$, so the integrals of the φ_ε are all 1. They are non-negative. Elementary estimates do show that, for fixed $\delta > 0$, $\int_{|x|\ge\delta}\varphi_\varepsilon \to 0$ as $\varepsilon \to 0^+$. This verifies that the φ_ε form an approximate identity in a slightly less-than-strictest sense, so the assertion holds. ///

Discussion: In contrast to the previous example, the functions $\varphi_n(x) = \frac{\sin 2\pi nx}{\pi x}$ (related to the Fourier-Dirichlet kernel) do *not* form an approximate identity in a straightforward sense, since they are not non-negative. And they are not in $L^1(\mathbb{R})$, so the integrals for their Fourier transforms do not converge absolutely. But they are in $L^2(\mathbb{R})$, so *do* have Fourier transforms in the extended Fourier-Plancherel sense, and the identity $\widehat{f * \varphi_n} = \widehat{f} \cdot \widehat{\varphi}_n$ still holds. By Fourier inversion, $\widehat{\varphi}_n = \chi_{[-t,t]}$. In particular, $\widehat{f} \cdot \chi_{[-t,t]}$ converges in $L^2(\mathbb{R})$ to $\widehat{f}$ (and $\widehat{f}$ is certainly in L^2, because it is in $\mathscr{S}$).

Plancherel shows that the Fourier(-Plancherel) map and inverse are isometric isomorphisms $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, so

$$f \;=\; (\,\widehat{f}\,)^\vee \;=\; \big(L^2 - \lim_n \widehat{f} \cdot \chi_{[-t,t]}\big)^\vee \;=\; L^2 - \lim_n \Big(\big(\widehat{f} \cdot \chi_{[-t,t]}\big)^\vee\Big)$$

$$=\; L^2 - \lim_n \Big((\,\widehat{f}\,)^\vee \;*\; \chi^\vee_{[-t,t]}\Big) \;=\; f * \frac{\sin 2\pi t x}{\pi x}$$

as claimed. ///