(*April 5, 2019*)

Examples discussion 05

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-disc-05.pdf]

[05.1] Show that multiplication by x, and also differentiation d/dx, are continuous operators $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

Discussion: By the definition of the topology on $\mathscr{S}$, and by the equivalent of continuity at 0 and continuity for linear maps, it suffices to show that, given $\varepsilon > 0$ and seminorm ν on $\mathscr{S}$, there is $\delta > 0$ and seminorm μ on Schwartz functions φ such that $\mu(\varphi) < \delta$ implies $\nu(\varphi) < \varepsilon$.

For $\nu(\varphi) = \sup_{0 \le i \le k} \sup_{x \in \mathbb{R}} (1+x^2)^n \cdot |f^{(i)}(x)|$, observe that

$$|(xf)^{(i)}(x)| \;=\; |(f + xf')^{(i-1)}(x)| \;=\; |(2f' + xf'')^{(i-2)}(x)| \;=\; \ldots \;=\; |(if^{(i-1)} + xf^{(i)})(x)|$$

$$\le\; |f^{(i-1)}(x)| + (1+x^2)\, f^{(i)})(x)|$$

by induction. Thus,

$$\nu(xf) \;=\; \sup_{0\le i\le k} \sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |f^{(i)}(x)| \;\le\; \sup_{0\le i\le k} \sup_{x\in\mathbb{R}} (1+x^2)^n |f^{(i-1)}(x)| + \sup_{x\in\mathbb{R}} |(1+x^2)^{n+1}\, f^{(i)})(x)|$$

$$\le\; 2 \sup_{0\le i\le k} \sup_{x\in\mathbb{R}} |(1+x^2)^{n+1}\, f^{(i)})(x)|$$

So if we make

$$\mu(f) \;=\; \sup_{0\le i\le k} \sup_{x\in\mathbb{R}} |(1+x^2)^{n+1}\, f^{(i)})(x)|$$

smaller than ε, then $|\nu(xf)| < 2\varepsilon$, giving the continuity of multiplication by x. ///

Even more simply, $|(f')^{(i)}(x)| = |f^{(i+1)}(x)|$ gives

$$\nu(f') \;=\; \sup_{0\le i\le k} \sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |(f')^{(i)}(x)| \;=\; \sup_{1\le i\le k+1} \sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |f^{(i)}(x)| \;\le\; \sup_{0\le i\le k+1} \sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |f^{(i)}(x)|$$

So if we make

$$\mu(f) \;=\; \sup_{0\le i\le k+1} \sup_{x\in\mathbb{R}} |(1+x^2)^{n+1}\, f^{(i)})(x)|$$

smaller than ε, then $|\nu(f')| < \varepsilon$. ///

[05.2] Show that d/dx is a continuous operator on $C^\infty(\mathbb{T})$, where $\mathbb{T}$ is the circle $\mathbb{R}/2\pi\mathbb{Z}$.

Discussion: This is simpler than the case of differentiation on Schwartz functions, since there is no issue about growth at infinity. For $\nu(f) = \sup_{0\le i\le k} \sup_{x\in\mathbb{T}} |f^{(i)}(x)|$,

$$\nu(f') \;=\; \sup_{0\le i\le k} \sup_{x\in\mathbb{T}} |(f')^{(i)}(x)| \;=\; \sup_{1\le i\le k+1} \sup_{x\in\mathbb{T}} |f^{(i)}(x)| \;\le\; \sup_{0\le i\le k+1} \sup_{x\in\mathbb{T}} |f^{(i)}(x)|$$

So if

$$\mu(f) \;=\; \sup_{0\le i\le k+1} \sup_{x\in\mathbb{R}} |f^{(i)})(x)|$$

is smaller than ε, then $|\nu(f')| < \varepsilon$, giving continuity. ///

[05.3] Show that $\delta(\varphi) = \varphi(0)$ is a tempered distribution.

Discussion: By the definition of the topology on $\mathscr{S}$, and by the equivalent of continuity at 0 and continuity for linear maps, it suffices to show that, given $\varepsilon > 0$, there is $\eta > 0$ and seminorm ν on Schwartz functions φ such that $\nu(\varphi) < \eta$ implies $|\varphi(0)| < \varepsilon$. This succeeds for $\nu(\varphi) = \sup_{x\in\mathbb{R}} |\varphi(x)|$ and $\eta = \varepsilon$. ///

[05.4] Let $\psi_n(x) = e^{inx}$. Show that $\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$ converges in the Sobolev space $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$.

Discussion: Using the *spectral characterization* of these Sobolev spaces, the question is for what $s \in \mathbb{R}$

$$\sum_{n\in\mathbb{Z}} 1^2 \cdot (1+n^2)^s \;<\; +\infty$$

where the 1^2 is the absolute value squared of the Fourier coefficients. By the integral test, or other elementary tests, the precise condition is that $s < -\frac{1}{2}$. ///

[05.5] Differentiate $\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$ twice.

Discussion: Let u be that generalized function, which we've shown is in $H^s(\mathbb{T})$ for every $s < -\frac{1}{2}$. Thus, we can distributionally differentiate u by differentiating the Fourier expansion termwise, since we have seen that (extended) differentiation is a continuous map $H^s \to H^{s-1}$ for all $s \in \mathbb{R}$:

$$u' \;=\; \sum_n in \cdot e^{inx} \qquad\qquad (\text{converging in } H^{s-1} \text{ with } s < -\tfrac{1}{2})$$

and

$$u'' \;=\; \sum_n (in)^2 \cdot e^{inx} \qquad\qquad (\text{converging in } H^{s-2} \text{ with } s < -\tfrac{1}{2})$$

That's all there is to it. ///

[05.6] Show that the principal value integral $\lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{f(x)}{x} dx$ is a tempered distribution, and satisfies $x \cdot u = 1$.

Discussion: Let u be that functional. For fixed $\varepsilon > 0$, integrate by parts, so, ignoring the boundary terms at infinity, hoping they're 0 (!?!), we still do definitely have boundary terms at $\pm\varepsilon$:

$$u(f) \;=\; \lim_{\varepsilon\to 0^+} \Big(\Big[\log|x| \cdot f(x) \Big] - \int_{|x|>\varepsilon} f'(x) \cdot \log|x| \; dx \Big)$$

The boundary terms

$$\log|\varepsilon| \cdot f(\varepsilon) \;-\; \log|-\varepsilon| \cdot f(-\varepsilon) \;=\; (2\varepsilon \cdot \log\varepsilon) \cdot \frac{f(\varepsilon) - f(-\varepsilon)}{2\varepsilon}$$

are 0: differentiability of f at 0 implies that

$$\lim_{\varepsilon\to 0^+} \frac{f(\varepsilon) - f(-\varepsilon)}{2\varepsilon} \;=\; f'(0)$$

and in particular the limit *exists*, while

$$\lim_{\varepsilon\to 0^+} \varepsilon \cdot \log\varepsilon \;=\; 0$$

Thus,

$$|u(f)| \;=\; \Big| \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} f'(x) \cdot \log|x| \; dx \Big| \;=\; \Big| \int_{\mathbb{R}} f'(x) \cdot \log|x| \; dx \Big|$$

Further,

$$\Big|\int_{\mathbb{R}} f'(x)(1+x^2)\cdot\frac{\log|x|}{1+x^2}\,dx\Big| \;\le\; \sup_{x\in\mathbb{R}}(1+x^2)|f'(x)|(1+x^2)\cdot\int_{\mathbb{R}}\frac{|\log|x||}{1+x^2}\,dx$$

The latter integral is a finite constant, so to make $|u(f)|$ small it suffices to make the seminorm

$$\mu(f) \;=\; \sup_{0\le i\le 1}\sup_{x\in\mathbb{R}}(1+x^2)|f^{(i)}(x)|$$

small, proving the continuity. ///

For $f\in\mathscr{S}$,

$$(x\cdot u)(f) \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon}\frac{x\cdot f(x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} f(x)\,dx \;=\; \int_{\mathbb{R}} f(x)\,dx \;=\; \int_{\mathbb{R}} 1\cdot f(x)\,dx \;=\; 1(\varphi)$$

thinking of 1 as the integrate-against-1 distribution, since φ is continuous at 0. Thus, $x\cdot u = 1$. ///

[05.7] Show that $\widehat{\delta}=1$ by approximating δ by Gaussians.

Discussion: We have already shown that $u_n(x)=\sqrt{n}\cdot e^{-\pi n x^2}$ is an *approximate identity*, meaning that

$$u_n(f) \;\longrightarrow\; f(0) \;=\; \delta(f)$$

for every $f\in\mathscr{S}$, which is exactly to say that $u_n\to\delta$ in the weak dual topology on $\mathscr{S}^*$. Fourier transform on tempered distributions is continuous, so Fourier transform and the weak-dual-topology limit can be interchanged:

$$\widehat{\delta} \;=\; \lim_n \widehat{u_n} \;=\; \lim_n e^{-\pi x^2/n}$$

from earlier computations of Fourier transforms of Gaussians. By Lebesgue dominated convergence, for $f\in L^1$,

$$\lim_n\int_{\mathbb{R}} e^{-\pi x^2/n}\cdot f(x)\,dx \;=\; \int_{\mathbb{R}}\lim_n e^{-\pi x^2/n}\cdot f(x)\,dx \;=\; \int_{\mathbb{R}} 1\cdot f(x)\,dx \;=\; 1(f) \qquad\text{(for all } f\in\mathscr{S})$$

That is, $\widehat{\delta}=1$. ///

[05.8] Show that $\lim_n \dfrac{1}{1+(x-n)^2}=0$ in $\mathscr{S}(\mathbb{R})^*$.

Discussion: It is implicit in the question that the functionals are *integrate against* the functions $\frac{1}{1+(x-n)^2}$. By definition of the weak dual topology on $\mathscr{S}^*$, we must show that for every $f\in\mathscr{S}(\mathbb{R})$

$$\lim_n\int_{\mathbb{R}}\frac{1}{1+(x-n)^2}\cdot f(x)\,dx \;=\; 0$$

The idea is that most of the mass of $1/(1+(x-n)^2)$ is centered around n, near which f is small for large n.

Given the convergence of the integral of $1/(1+x^2)$, for every $\varepsilon>0$ there is N such that

$$\int_{|x|\ge N}\frac{1}{1+x^2}\,dx \;<\; \varepsilon$$

and then by changing variables, for every n,

$$\int_{|x-n|\ge N}\frac{1}{1+(x-n)^2}\,dx \;<\; \varepsilon$$

For each $f \in \mathscr{S}$,

$$\Big|\int_{\mathbb{R}} \frac{1}{1+(x-n)^2}\cdot f(x)\,dx\Big| \;=\; \Big|\int_{\mathbb{R}} \frac{1}{(1+(x-n)^2)(1+x^2)}\cdot(1+x^2)f(x)\,dx\Big| \;\le\; \int_{\mathbb{R}} \frac{1}{(1+(x-n)^2)(1+x^2)}\,dx\cdot\sup_{x\in\mathbb{R}}(1+x^2)|\,f(x)|$$

Thus, it suffices to show that

$$\int_{\mathbb{R}} \frac{1}{(1+(x-n)^2)(1+x^2)}\,dx \;\longrightarrow\; 0$$

Indeed,

$$\int_{\mathbb{R}} \frac{1}{(1+(x-n)^2)(1+x^2)}\,dx \;=\; \int_{|x|\le n/2} \frac{1}{(1+(x-n)^2)(1+x^2)}\,dx \;+\; \int_{|x|>n/2} \frac{1}{(1+(x-n)^2)(1+x^2)}\,dx$$

$$\le\; \int_{|x|\le n/2} \frac{1}{((\frac{n}{2})^2)(1+x^2)}\,dx \;+\; \int_{|x|>n/2} \frac{1}{(1+(x-n)^2)(1+(\frac{n}{2})^2)}\,dx$$

$$=\; \frac{4}{n^2}\int_{|x|\le n/2} \frac{1}{1+x^2}\,dx \;+\; \frac{4}{n^2}\int_{|x|>n/2} \frac{1}{1+(x-n)^2}\,dx \;=\; \frac{8}{n^2}\int_{|x|\le n/2} \frac{1}{1+x^2}\,dx$$

$$\le\; \frac{8}{n^2}\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx \;\longrightarrow\; 0$$

since the last integral is just a constant. This proves that $1/(1+(x-n)^2) \to 0$ in the weak dual topology on $\mathscr{S}^*$. ///

[05.9] Determine the constant c such that $x^2\delta'' = c\cdot\delta$.

Discussion: Compute directly: for $f \in \mathscr{S}$,

$$(x^2\delta'')(f) \;=\; \delta''(x^2\cdot f) \;=\; -\delta'(2x\cdot f + x^2\cdot f') \;=\; \delta(2\cdot f + 4x\cdot f' + x^2 f'')$$

$$=\; 2\cdot f(0) + 4\cdot 0\cdot f'(0) + 0^2\cdot f''(0) \;=\; 2\cdot f(0) \;=\; 2\delta(f)$$

So $x^2\delta'' = 2\delta$. ///

[05.10] Show that the characteristic function of an interval is in $H^{\frac12-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac12}(\mathbb{R})$.

Discussion:

[05.11] Show that $f(x) = e^{-|x|}$ is in $H^{1-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^1(\mathbb{R})$.

[05.12] Show that $\sin(nx) \to 0$ in the $\mathscr{S}^*$-topology as $n \to +\infty$.

[05.13] Show that the (distributional) derivative of a positive, regular Borel measure μ on $\mathbb{R}$ is in $H^{-\frac12-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$. (Hint: use Sobolev imbedding and Riesz-Markov-Kakutani theorem.)
