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Examples discussion 06

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-disc-06.pdf]

[06.1] Let $\psi_n(x) = e^{2\pi inx}$. Let $\delta_\mathbb{Z}$ be the *Dirac comb*, that is, a periodic version of Dirac's δ, describable as having Fourier series

$$\delta_\mathbb{Z} \;=\; \sum_{n\in\mathbb{Z}} 1\cdot\psi_n \qquad\qquad \text{(converging in } H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \text{ for all } \varepsilon > 0)$$

With $\lambda \notin \mathbb{R}$, show that the differential equation

$$u'' - \lambda\cdot u \;=\; \delta_\mathbb{Z}$$

has a periodic solution $u \in H^{\frac{3}{2}-\varepsilon}(\mathbb{T}) \subset C^o(\mathbb{T})$, using Fourier series, *by division*. Show that the equation $v'' - \lambda v = f$ is solved by

$$v \;=\; \int_\mathbb{T} u(x-t)\, f(t)\; dt \;=\; \int_0^1 u(x-t)\, f(t)\; dt$$

Discussion: Let's assume that we are asking for a solution u that is at worst a tempered distribution. Thus, we can take Fourier transform, obtaining

$$(4\pi^2\xi^2 - \lambda)\widehat{u} \;=\; \widehat{\delta} \;=\; 1$$

Obviously we want to *divide* by $4\pi^2\xi^2-\lambda$. Unlike some other examples, where division is not quite legitimate, here, we can achieve the effect of division by *multiplication* by the smooth, bounded function $1/(4\pi^2\xi^2+\lambda)$, since $4\pi^2\xi^2+\lambda$ does not vanish on $\mathbb{R}$. Thus,

$$\widehat{u} \;=\; \frac{1}{4\pi^2\xi^2 - \lambda}$$

Since the right-hand side is luckily in $L^1(\mathbb{R})$, we can compute its image under Fourier inversion by the literal integral, its inverse Fourier transform will be a continuous function (by Riemann-Lebesgue), so has meaningful pointwise values:

$$u(x) \;=\; \int_\mathbb{R} \frac{e^{2\pi i\xi x}}{(2\pi i\xi)^2 - \lambda}\; d\xi$$

The integral can be evaluated by *residues*: depending on the sign of the real part of $x\sqrt{\lambda}$, we use an auxiliary arc in the upper or lower half plane, so that $\xi \to e^{2\pi i\sqrt{\lambda}\xi x}$ is *bounded* in that half-plane. Thus, we pick up either $2\pi i$ times the residue at $\xi = \pm\sqrt{\lambda}/2\pi i$, or the negative (because the orientation is negative) of the residue at $\xi = \pm\sqrt{\lambda}/2\pi i$. Summarizing the two computations, this is

$$2\pi i\cdot\frac{e^{2\pi i\cdot(\sqrt{\lambda}/2\pi i)\cdot x}}{4\pi^2\cdot(\frac{\sqrt{\lambda}}{2\pi i} - \frac{-\sqrt{\lambda}}{2\pi i})} \;=\; \frac{-e^{\pm\sqrt{\lambda}|x|}}{2\sqrt{\lambda}}$$

with sign chosen so that the function is bounded. This answers the first part.

To see that this function u has the property of *fundamental solution*, as indicated, we (at least) heuristically compute

$$(\Delta-\lambda)v \;=\; (\Delta-\lambda)\int_\mathbb{T} u(x-t)\, f(t)\; dt \;=\; \int_\mathbb{T} (\Delta_x-\lambda)u(x-t)\, f(t)\; dt \;=\; \int_\mathbb{T} (u''-\lambda u)(x-t)\, f(t)\; dt$$

since Δ is translation-invariant. Then this would be

$$\int_{\mathbb{T}} \delta(x-t)\, f(t)\, dt \;=\; f(x)$$

as desired. The work remaining is to justify moving the differential operator inside the integral, and to understand how (after doing that) the literal integral has to be something more abstract, to make sense of supposedly integrating against Dirac δ. We'll do this in notes, elsewhere. ///

[06.2] Show that $u'' = \delta_{\mathbb{Z}}$ has no solution on the circle $\mathbb{T}$. (*Hint:* Use Fourier series.) Show that $u'' = \delta_{\mathbb{Z}} - 1$ *does* have a solution.

Discussion: In Fourier series converging in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ for all $\varepsilon > 0$, $\delta_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}} 1\cdot\psi_n$, where $\psi_n(x) = e^{2\pi i n x}$. A function u in the relatively large-yet-tractable space $H^{-\infty}(\mathbb{T})$ has a Fourier expansion $u = \sum_n \widehat{u}(n)\cdot\psi_n$. Application of the (extended-sense) second derivative operator can be done termwise (by design), and annihilates the $n=0$ term. That is, no u'' can have 0^{th} Fourier coefficient 1, as does $\delta_{\mathbb{Z}}$, so that equation is not solvable. ///

In contrast, $\delta_{\mathbb{Z}} - 1$ has exactly lost that difficult Fourier component, and, in terms of Fourier series, $u'' = \delta_{\mathbb{Z}} - 1$ is

$$\sum_{n\in\mathbb{Z}} (2\pi i n)^2)\cdot \widehat{u}(n)\cdot\psi_n \;=\; \sum_{n\neq 0} 1\cdot\psi_n$$

has the solution *by division*

$$u \;=\; \sum_{n\neq 0} \frac{1}{(2\pi i n)^2}\psi_n$$

[06.3] On the circle $\mathbb{T}$, show that $u'' = f$ has a unique solution u orthogonal to the constant function, for all $f \in L^2(\mathbb{T})$ orthogonal to the constant function 1.

Discussion: We use Fourier series for functions in $H^{-\infty}(\mathbb{T})$: we want to solve the equation

$$\sum_n \widehat{f}(n)\cdot\psi_n \;=\; f \;=\; u'' \;=\; \Delta\sum_n \widehat{u}(n)\cdot\psi_n$$

First, the differentiation can be moved inside the sum, that is, we can differentiate termwise, because we have shown that differentiation is a continuous map $H^s \to H^{s-1}$ for all s, by design. That is, since sums are limits of their finite partial sums, and since

$$\frac{d}{dx}(H^s - \lim u_n) \;=\; H^{s-1} - \lim \frac{d}{dx} u_n$$

we have justified termwise differentiation. By the very convenient mutual orthogonality of the exponentials ψ_n in all the Hilbert spaces H^s, we have uniqueness of Fourier coefficients in $H^{-\infty}$, even though the latter is only a union of Hilbert spaces, and not a Hilbert space itself. Thus, for $u'' = f$ to have a solution, it must be that

$$((2\pi i n)^2 - \lambda)\cdot\widehat{u}(n) \;=\; \widehat{f}(n)$$

This is impossible if $\widehat{f}(0) \neq 0$, since the coefficient on the left is 0, so we must require $\widehat{f}(0) = 0$, which (in whatever Hilbert space H^s lies) is to say that $f \perp 1$.

Given $\widehat{f}(0) = 0$, we can solve the equation by division. But there is ambiguity of $\widehat{u}(0)$. For uniqueness, we also require $\widehat{u}(0) = 0$, which, again, is $u \perp 1$ in whatever Hilbert space u lies. ///

[06.4] Compute $\widehat{\cos x}$.

Discussion: This reduces to knowing that $\widehat{\delta} = 1$ and the behavior of Fourier transforms under translation... *[... iou ...]*

[06.5] Smooth functions $f \in \mathcal{E}$ act on distributions $u \in \mathcal{D}(\mathbb{R})^*$ by a dualized form of pointwise multiplication: $(f \cdot u)(\varphi) = u(f\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Show that if $x \cdot u = 0$, then u is *supported at* 0, in the sense that for $\varphi \in \mathcal{D}$ with $\operatorname{spt} \varphi \not\ni 0$, necessarily $u(\varphi) = 0$. Thus, by the theorem classifying such distributions, u is a linear combination of δ and its derivatives. Show that in fact $x \cdot u = 0$ implies that u is a multiple of δ itself.

Discussion: For $\varphi \in \mathcal{D}$ whose support does *not* include 0, the function $1/x$ is defined and smooth on $\operatorname{spt} \varphi$. Thus, $x \to \varphi(x)/x$ is in $\mathcal{D}$. For such φ,

$$u(\varphi) \;=\; u(x \cdot \frac{\varphi}{x}) \;=\; 0$$

Thus, $\operatorname{spt} u = \{0\}$, so by the theorem is a finite linear combination $u = \sum_{i=0}^n c_i\, \delta^{(i)}$ with scalars c_i. To see that in fact only δ itself can appear, we use the idea that $1, x, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots, \frac{x^n}{n!}$ are essentially a *dual basis* to $\delta, \delta', \delta'', \ldots, \delta^{(n)}$. One way to make this completely precise is to use a smooth cut-off function $\eta \in \mathcal{D}$ around 0, namely, identically 1 on a neighborhood of 0. Then $\eta \cdot x^i \in \mathcal{D}$, and

$$\delta^{(i)}(\eta \cdot \frac{x^j}{j!}) \;=\; \begin{cases} 1 & (\text{for } i = j) \\ \\ 0 & (\text{for } i \neq j) \end{cases}$$

In particular, this shows that the derivatives of δ are *linearly independent*. For $0 \le j \in \mathbb{Z}$,

$$0 \;=\; (x \cdot u)(x^j) \;=\; (x \cdot \sum_i c_i\, \delta^{(i)})(x^j) \;=\; \sum_i c_i\, \delta^{(i)}(x \cdot x^j) \;=\; \sum_i c_i\, \delta^{(i)}(x^{j+1}) \;=\; (j+1)! \cdot c_{j+1}$$

Thus, $c_j = 0$ for $j \ge 1$, and u is a multiple of δ itself. ///

[06.6] Given f in the Schwartz space $\mathscr{S}$, show that there is $F \in \mathscr{S}$ with $F' = f$ if and only if $\int_{\mathbb{R}} f = 0$.

Discussion: On one hand, if $f = F'$ for $F \in \mathscr{S}$, then $\int_{-\infty}^x f(y)\, dy = F(x)$. Since $\lim_{x \to +\infty} F(x) = 0$, $\int_{\mathbb{R}} f = 0$.

On the other hand, if $\int_{\mathbb{R}} f = 0$, let $F(x) = \int_{-\infty}^x f$, and show that $F \in \mathscr{S}$. Since $F' = f$ by the fundamental theorem of calculus, the (higher) derivatives of F are those of f, so all that needs to be shown is that F itself is of rapid decay. For $x \to -\infty$,

$$|F(x)| \;\le\; \int_{-\infty}^x |f| \;\le\; \int_{-\infty}^x |1+y^2|^{-N} \cdot \sup_{t \in \mathbb{R}} \Big|(1+t^2)^N \cdot f(t)\Big|\, dy \;\le\; \sup_{t \in \mathbb{R}} \Big|(1+t^2)^N \cdot f(t)\Big| \cdot \int_{-\infty}^x |1+y^2|^{-N}\, dy$$

$$\le\; \sup_{t \in \mathbb{R}} \Big|(1+t^2)^N \cdot f(t)\Big| \cdot \int_{-\infty}^x \frac{dt}{t^N} \;\le\; \sup_{t \in \mathbb{R}} \Big|(1+t^2)^N \cdot f(t)\Big| \cdot \frac{1}{|x|^{N-1}} \qquad (\text{for } x \to -\infty)$$

giving the rapid decay. For $x \to +\infty$, using the condition $\int_{\mathbb{R}} f = 0$,

$$F(x) \;=\; \int_{-\infty}^x f \;=\; \int_{\mathbb{R}} f - \int_x^\infty f \;=\; 0 - \int_x^\infty f$$

so for $x \to +\infty$ it suffices to similarly estimate

$$\Big|\int_x^\infty f\Big| \;\le\; \int_x^\infty (1+y^2)^{-N} \cdot \sup_{t \in \mathbb{R}} |(1+t^2)^N \cdot f(t)|\, dy \;\le\; \sup_{t \in \mathbb{R}} |(1+t^2)^N \cdot f(t)| \cdot \int_x^\infty (1+y^2)^{-N}\, dy$$

which similarly gives the rapid decay as $x \to +\infty$. ///

[06.7] Let $u(x) = e^x \cdot \sin(e^x)$. Explain in what sense the integral $\int_{\mathbb{R}} f(x)\, u(x)\, dx$ converges for every $f \in \mathscr{S}$.

Discussion: The idea is to integrate by parts, noting that $u = v'$ with $v(x) = \cos(e^x)$. We must be careful with the boundary terms:

$$\int_{\mathbb{R}} f(x)\, u(x)\, dx \;=\; \int_{\mathbb{R}} f(x)\, v'(x)\, dx \;=\; \lim_{M,N\to+\infty} \int_{-M}^{N} f(x)\, v'(x)\, dx$$

$$=\; \lim_{M,N\to+\infty} \Big(\Big[f(x)\, v(x) \Big]_{-M}^{N} - \int_{-M}^{N} f'(x)\, v(x)\, dx \Big)$$

Since $v(x)$ is bounded and f' is of rapid decay, the limit *exists*, so the original integral is convergent. Further, the value is correctly determined by integration by parts, namely

$$-\int_{-\infty}^{\infty} f'(x)\, v(x)\, dx \;=\; -\int_{-\infty}^{\infty} f'(x)\, \cos(e^x)\, dx$$

That is, for $f \in \mathscr{S}$ and functions such as u obtained by differentiating bounded smooth functions, integration by parts is completely justifiable via the natural estimates. ///

[06.8] Compute the Fourier transform of the sign function

$$\mathrm{sgn}(x) \;=\; \begin{cases} 1 & (\text{for } x > 0) \\ -1 & (\text{for } x < 0) \end{cases}$$

Hint: $\frac{d}{dx}\mathrm{sgn} \;=\; 2\delta$. Since Fourier transform converts d/dx to multiplication by $2\pi i x$, this implies that $(2\pi i)x \cdot \widehat{\mathrm{sgn}} = 2\widehat{\delta} = 2$. Thus, $(\pi i)x \cdot \widehat{\mathrm{sgn}} = 1$.

Discussion: From the hint, $x \cdot (\pi i\, \widehat{\mathrm{sgn}}) = 1$. Also, we have seen that the principal-value functional u satisfies $x \cdot u = 1$: for all $\varphi \in \mathscr{S}$,

$$(x\cdot u)(\varphi) \;=\; u(x\cdot\varphi) \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{x \cdot \varphi(x)}{x}\, dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \varphi(x)\, dx \;=\; \int_{\mathbb{R}} \varphi(x)\, dx \;=\; \int_{\mathbb{R}} 1\cdot\varphi(x)\, dx \;=\; 1(\varphi)$$

Thus,

$$x \cdot (u - \pi i\, \widehat{\mathrm{sgn}}) \;=\; 0$$

We have also seen that $x \cdot v = 0$ for distribution v implies that v is a constant multiple of δ. Thus, $u - \pi i\, \widehat{\mathrm{sgn}}$ is a multiple of δ. In fact, the multiple is 0, because δ is *even*, while u, sgn, and thus $\widehat{\mathrm{sgn}}$, are all *odd*. [1] That is, $\widehat{\mathrm{sgn}} \;=\; \frac{1}{\pi i} u$. ///

[06.9] On $\mathbb{R}^n$, show that $|x|^2 \cdot \Delta\delta = 2n \cdot \delta$.

Discussion: Another direction computation, using the duality characterization: for $\varphi \in \mathscr{S}$,

$$(r^2 \Delta\delta)(\varphi) \;=\; (\Delta\delta)(r^2\varphi) \;=\; =\; (-1)^2 \delta(\Delta(r^2\varphi))$$

Compute

$$\Delta(r^2\varphi) \;=\; \sum_i \frac{\partial^2}{\partial x_i^2}(r^2\varphi) \;=\; \sum_i \frac{\partial}{\partial x_i}(2x_i\varphi + r^2 \frac{\partial\varphi}{\partial x_i})$$

[1] This notion of parity can be defined for distributions from the obvious notion for functions $(\theta \cdot f)(x) = f(-x)$, and then $(\theta \cdot v)(f) = v(\theta \cdot f)$ for distributions v.

$$= \sum_i 2\varphi + 2x_i \frac{\partial\varphi}{\partial x_i} + r^2 \frac{\partial^2\varphi}{\partial x_i^2} \;=\; 2n\varphi + \sum_i 2x_i \frac{\partial\varphi}{\partial x_i} + nr^2\Delta\varphi$$

Applying δ to this gives

$$2n\varphi(0) + \sum_i 2\cdot 0\cdot \frac{\partial\varphi}{\partial x_i}(0) + n\cdot 0\cdot(\Delta\varphi)(0) \;=\; 2n\varphi(0) \;=\; 2n\delta(\varphi)$$

as claimed. ///

[06.10] On $\mathbb{R}^2$, compute the Fourier transform of $(x\pm iy)^n \cdot e^{-\pi(x^2+y^2)}$ for $n=0,1,2,\ldots$. (*Hint:* Re-express things, including Fourier transform, in terms of $z=x+iy$ and $\overline{z}=x-iy$, $w=u+iv$, and $\overline{w}=u-iv$.)

Discussion: Using z and w, the functions are $z^n e^{-\pi z\overline{z}}$ and $\overline{z}^n e^{-\pi z\overline{z}}$, and Fourier transform is

$$\int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, z^n e^{-\pi z\overline{z}}\, dx\, dy \;=\; \int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, \frac{1}{(-\pi)^n}\Big(\frac{\partial}{\partial\overline{z}}\Big)^n e^{-\pi z\overline{z}}\, dx\, dy$$

Imagining that we can integrate by parts, this is

$$(-1)^n \frac{1}{(-\pi)^n}\int_{\mathbb{R}^2}\Big(\frac{\partial}{\partial\overline{z}}\Big)^n e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy \;=\; \frac{1}{\pi^n}\int_{\mathbb{R}^2}(-\pi i w)^n e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy$$

$$= (-i)^n\, w^n \int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy \;=\; i^{-n}\, w^n\, e^{-\pi(w\overline{w})}$$

since we know the Fourier transform of a Gaussian. A similar computation with roles of $z,\overline{z}$ reversed accomplishes the other computation. That is, $(x\pm iy)^n e^{-\pi(x^2+y^2)}$ is an eigenfunction for Fourier transform, with eigenvalue $i^{-|n|}$. ///

[06.11] Show that on $\mathbb{R}^n$ with $n\geq 3$,

$$\Delta\frac{1}{|x|^{n-2}} \;=\; \text{constant multiple of } \delta$$

That is, up to a constant, $1/|x|^{n-2}$ is a *fundamental solution* for the Laplacian.

[... iou ...]

[06.12] In the context of complex analysis, the Cauchy-Riemann operator is

$$\frac{\partial}{\partial\overline{z}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

The Cauchy-Riemann equation characterizing holomorphic functions f is

$$\frac{\partial}{\partial\overline{z}} f \;=\; 0$$

Show that

$$\frac{\partial}{\partial\overline{z}}\frac{1}{z} \;=\; \text{constant multiple of } \delta$$

That is, $1/z$ is a *fundamental solution* for the Cauchy-Riemann operator. (So the shape of the Cauchy integral formula is perhaps not so surprising.)

[... iou ...]

[06.13] Show that, given a distribution u on $\mathbb{T}^n$, for any $0 \leq k \in \mathbb{Z}$ there is $f \in C^k(\mathbb{T}^n)$ and sufficiently large ℓ such that $(1-\Delta)^n f = u$.

Done in class.

[06.14] Show that, given a compactly-supported distribution u on $\mathbb{R}^n$, for any $0 \leq k \in \mathbb{Z}$ there is $f \in C^k(\mathbb{R}^n)$ and sufficiently large ℓ such that $(1-\Delta)^\ell f = u$.

Done in class.
