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Review examples 00

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-ex-00.pdf]

If you want feedback on your write-ups on any of these examples, please get me your write-ups by Friday, 14 Sept, 2018.

[00.1] (There is not much hope to make sense of the outcome of an uncountable number of non-zero operations:) Let Ω be an *uncountable* collection of positive real numbers. Letting F range over all finite subsets of Ω, show that $\sup_F \sum_{\alpha\in F} \alpha = +\infty$.

[00.2] Prove (or review the proof) that a continuous real-valued function f on a finite closed interval $[a,b] \subset \mathbb{R}$ is *uniformly* continuous: for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in [a,b]$, $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

[00.3] Prove (or review the proof) that a *uniform* pointwise limit of continuous, real-valued functions on $[a,b]$ is continuous.

[00.4] Prove (or review the proof) of the *Fundamental Theorem of Calculus*: for a *continuous* function f on $[a,b]$, the function $F(x) = \int_a^x f(t)\,dt$ is *continuously differentiable*, and has derivative f. (Use Riemann's integral.)

[00.5] Prove (or review the proof) that for a sequence of real-valued functions f_n on $[0,1]$ approaching f *uniformly* pointwise, $\lim_n \int_0^1 f_n(x)\,dx = \int_0^1 \lim_n f_n(x)\,dx$. (Use Riemann's integral.)

[00.6] Show that every open subset of $\mathbb{R}$ is a *countable* union of open intervals.

[00.7] Define *Lebesgue (outer) measure* $\mu(E)$ of subsets E of $\mathbb{R}$ by

$$\mu(E) \;=\; \inf\{\sum_{n=1}^{\infty} |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where M is Cantor's middle-thirds set.
