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Examples 01

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2017-18/real-ex-01.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 05 Oct, 2018.

[01.1] Show that the closed unit ball in ℓ^2, although *closed* and *bounded*, is *not compact*, by showing it is not *sequentially* compact.

[01.2] Show that the closed unit ball in $C^o[a, b]$ is not compact, despite being closed and bounded.

[01.3] Let X be a metric space with a countable dense subset D. Show that every open set in X is a countable union of open balls.

[01.4] Let X be a compact metric space. Show that a continuous function on X is *uniformly* continuous.

[01.5] Let X be a compact metric space. Show that a uniform pointwise limit of continuous real-valued functions is continuous.

[01.6] Show that $C^o[a, b]$ is not complete with the $L^2[a, b]$ metric.

[01.7] Show that $C^1[a, b]$ is not complete with the $C^o[a, b]$ metric.

[01.8] Show that $C^1[a, b]$ is complete, with the $C^1[a, b]$ metric

$$d(f, g) \;=\; \sup_{a \le x \le b} |f(x) - g(x)| + \sup_{a \le x \le b} |f'(x) - g'(x)|$$

[01.9] Show that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals r_n,

$$C(r) \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le r_n\}$$

is compact if and only if $\sum_n |r_n|^2 < \infty$. (Hint: use the total boundedness criterion.)

[01.10] (Originally, this was formulated about more general metric spaces, but that introduced complications not of interest to us, and maybe was not quite correct as stated *anyway*.) Let $|\cdot|_1$ and $|\cdot|_2$ be two norms on a real or complex vector space X. Suppose that $|x|_1 \ge |x|_2$ for all $x \in X$. Let X_i be the completion of X with respect to the metric associated to $|\cdot|_i$. Show that the identity map $X \to X$ *extends by continuity* to a continuous *injection* $X_1 \to X_2$.

Comment: In fact, although the general extension-by-continuity works fine, the injectivity is most reasonably verified by using some non-trivial results on Banach spaces... which we've not covered yet. So maybe best to ignore this question for the time being, apart from seeing that it's harder-than-it-looks to verify injectivity in a completely elementary way.