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Examples 02

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2018-19/real-ex-02.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, October 24, 2018.

[02.1] The space of continuous functions on $\mathbb{R}$ *going to* 0 *at infinity* is

$$C^o_o(\mathbb{R}) \;=\; \{f \in C^o(\mathbb{R}) : \text{for every } \varepsilon > 0 \text{ there is } T \text{ such that } |f(x)| < \varepsilon \text{ for all } |x| \ge T\}$$

Show that the closure of $C^o_c(R)$ in the space $C^o_{\text{bdd}}(\mathbb{R})$ of bounded continuous functions with sup norm, is $C^o_o(\mathbb{R})$.

[02.2] Show that $|\int_a^b f|^2 \le |b-a| \cdot \int_a^b |f|^2$.

[02.3] In ℓ^2, show that the unique point in the closed unit ball closest to a point v *not* inside that ball is $v/|v|_{\ell^2}$.

[02.4] One form of the *sawtooth* function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\widehat{f}(n)$. From Plancherel-Parseval's theorem for this function, show that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots \;=\; \frac{\pi^2}{6}$$

[02.5] Show that there is *no* $f_o \in C^o[0,1]$ so that, for all $g \in C^o[0,1]$, $\int_0^1 f_o(x)\, g(x)\, dx\rangle = g(\frac{1}{2})$.

[02.6] For $c_1 > c_2 > c_3 > \ldots > 0$ a monotone-decreasing sequence of positive reals, with $\lim_n c_n = 0$, show that, for every $0 < x < 2\pi$, $\sum_n c_n\, e^{inx}$ converges.

[02.7] Let $b = \{b_n\}$ be a sequence of complex numbers, such that there is a bound B such that, for every $c = \{c_n\} \in \ell^2$, $|\sum_n b_n c_n| \le B \cdot |c|_{\ell^2}$. Show that $b \in \ell^2$.

[02.8] For a vector subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the topological closure of W.

[02.9] Find two *dense* vector subspaces X, Y of ℓ^2 such that $X \cap Y = \{0\}$. (And, if you need further entertainment, can you find countably-many dense vector subspaces X_n such that $X_m \cap X_n = \{0\}$ for $m \ne n$?)
