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Examples 03

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-ex-03.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, November 28, 2018.

[03.1] Show that the characteristic function of a measurable set is a measurable function.

[03.2] For measurable $E \subset [0, 2\pi]$, show that $\lim_n \int_E e^{-inx}\,dx = 0$ as $n \to \infty$ ranging over integers.

[03.3] For $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, show that there is a constant C (depending on f) such that

$$\Big|\int_{t-\delta}^{t+\delta} f(x)\,dx\Big| \;<\; C \cdot \sqrt{\delta}$$

[03.4] For $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, show that, given $\varepsilon > 0$, there $\delta > 0$ such that

$$\Big|\int_{t-\delta}^{t+\delta} f(x)\,dx\Big| \;<\; \varepsilon$$

[03.5] For non-negative real-valued f, show that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\,dx \;=\; \int_{\mathbb{R}} f(x)\,dx$$

(whether or not the integrals are finite).

[03.6] For $f \in L^1(\mathbb{R})$, show that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\,dx \;=\; \int_{\mathbb{R}} f(x)\,dx$$

[03.7] (*Comparing L^p spaces*) Let $1 \le p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

[03.8] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\le\; w_1 a_1 + \ldots + w_n a_n$$

[03.9] (*Collecting Fourier transform pairs*) Compute the Fourier transforms of

$$\chi_{[a,b]} \qquad\qquad e^{-\pi x^2} \qquad\qquad f(x) = \begin{cases} e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x \le 0) \end{cases}$$

[03.10] Compute $\displaystyle\int_{\mathbb{R}} \Big(\frac{\sin x}{x}\Big)^2 dx$. (*Hint:* do not attempt to do this directly, nor by complex analysis.)

[03.11] Let $E \subset \mathbb{R}$ be a measurable set with finite measure. Show that $\displaystyle\int_E \cos(tx)\,dx \to 0$ as $t \to +\infty$.
