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Examples 04

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2018-19/real-ex-04.pdf]

For feedback on these examples, please get your write-ups to me by Friday, February 08, 2019.

[04.1] With $g(x) = f(x + x_o)$, express $\widehat{g}$ in terms of $\widehat{f}$, for $f \in L^1(\mathbb{R}^n)$.

[04.2] Let $\{b_n\}$ be a sequence of complex numbers. Suppose that, for every $\{a_n\} \in \ell^2$, $\sum_n a_n b_n$ converges. Show that $\{b_n\} \in \ell^2$.

[04.3] Let g be a measurable $[0, +\infty]$-value function on $[a, b]$ such that, for every $f \in L^2[a, b]$, $\int_a^b |f(x)\, g(x)|\, dx < \infty$. Show that $g \in L^2[a, b]$.

[04.4] Give a *persuasive* proof that the function

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this to make a *smooth step function*: 0 for $x \le 0$ and 1 for $x \ge 1$, and goes monotonically from 0 to 1 in the interval $[0, 1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1, 2, 3, \ldots\}$: for each n, $f_n(x) = 1$ for $x \in [-n, n]$, $f_n(x) = 0$ for $x \notin [-(n + 1), n + 1]$, and f_n goes monotonically from 0 to 1 in $[-(n + 1), -n]$ and monotonically from 1 to 0 in $[n, n + 1]$.

[04.5] Give an explicit non-zero function f such that $\int_{\mathbb{R}} x^n\, f(x)\, dx = 0$, for all $n = 0, 1, 2, \ldots$.

[04.6] Show that $\chi_{[a,b]} * \chi_{[c,d]}$ is a piecewise-linear function, and express it explicitly.

[04.7] Compute $e^{-\pi x^2} * e^{-\pi x^2}$ and $\dfrac{\sin x}{x} * \dfrac{\sin x}{x}$. (Be careful what you say: $\frac{\sin x}{x}$ is not in $L^1(\mathbb{R})$, so there are potential problems with convolution.)

[04.8] For $f \in \mathscr{S}$, show that

$$\lim_{\varepsilon \to 0^+} f(x) * \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} \;=\; f(x)$$

[04.9] For $f \in \mathscr{S}$, show that

$$\lim_{t \to +\infty} f(x) * \frac{2 \sin tx}{tx} \;=\; f(x)$$

[04.10] Evaluate the *Borwein integral*

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\, dx$$