(*February 19, 2019*)

Examples 05

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2018-19/real-ex-05.pdf]

For feedback on these examples, please get your write-ups to me by Friday, February 08, 2019.

[05.1] Show that multiplication by x, and also differentiation d/dx, are continuous operators $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

[05.2] Show that d/dx is a continuous operator on $C^\infty(\mathbb{T})$, where $\mathbb{T}$ is the circle $\mathbb{R}/2\pi\mathbb{Z}$.

[05.3] Show that $\delta(\varphi) = \varphi(0)$ is a tempered distribution.

[05.4] Let $\psi_n(x) = e^{inx}$. Show that $\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$ converges in the Sobolev space $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$.

[05.5] Differentiate $\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$ twice.

[05.6] Show that the principal value integral $\lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{f(x)}{x} dx$ is a tempered distribution, and satisfies $x \cdot u = 1$.

[05.7] Show that $\widehat{\delta} = 1$ by approximating δ by Gaussians.

[05.8] Show that $\lim\limits_n \dfrac{1}{1+(x-n)^2} = 0$ in $\mathscr{S}(\mathbb{R})^*$.

[05.9] Determine the constant c such that $x^2\delta'' = c \cdot \delta$.

[05.10] Show that the characteristic function of an interval is in $H^{\frac{1}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{1}{2}}(\mathbb{R})$.

[05.11] Show that $f(x) = e^{-|x|}$ is in $H^{\frac{3}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{3}{2}}(\mathbb{R})$.

[05.12] Show that $\sin(nx) \to 0$ in the $\mathscr{S}^*$-topology as $n \to +\infty$.

[05.13] Show that the (distributional) derivative of a *finite* positive, regular Borel measure μ on $\mathbb{T}$ is in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ for every $\varepsilon > 0$. (Hint: Sobolev imbedding and the context of the Riesz-Markov-Kakutani theorem.)
