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Examples 06

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-ex-06.pdf]

For feedback on these examples, please get your write-ups to me by Friday, March 29, 2019.

[06.1] Let $\psi_n(x) = e^{2\pi i n x}$. Let $\delta_{\mathbb{Z}}$ be the *Dirac comb*, that is, a periodic version of Dirac's δ, describable as having Fourier series

$$\delta_{\mathbb{Z}} \;=\; \sum_{n\in\mathbb{Z}} 1\cdot\psi_n \qquad\qquad \text{(converging in } H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \text{ for all } \varepsilon>0\text{)}$$

With $\lambda \notin \mathbb{R}$, show that the differential equation

$$u'' - \lambda\cdot u \;=\; \delta_{\mathbb{Z}}$$

has a periodic solution $u \in H^{\frac{3}{2}-\varepsilon}(\mathbb{T}) \subset C^o(\mathbb{T})$, using Fourier series, *by division*. Show that the equation $v''-\lambda v = f$ is solved by

$$v \;=\; \int_{\mathbb{T}} u(x-t)\,f(t)\,dt \;=\; \int_0^1 u(x-t)\,f(t)\,dt$$

[06.2] Show that $u'' = \delta_{\mathbb{Z}}$ has no solution on the circle $\mathbb{T}$. (*Hint:* Use Fourier series.) Show that $u'' = \delta_{\mathbb{Z}} - 1$ *does* have a solution.

[06.3] On the circle $\mathbb{T}$, show that $u'' = f$ has a unique solution for all $f \in L^2(\mathbb{T})$ orthogonal to the constant function 1.

[06.4] Compute $\widehat{\cos x}$.

[06.5] Smooth functions $f \in \mathcal{E}$ act on distributions $u \in \mathcal{D}(\mathbb{R})^*$ by a dualized form of pointwise multiplication: $(f\cdot u)(\varphi) = u(f\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Show that if $x\cdot u = 0$, then u is *supported at* 0, in the sense that for $\varphi \in \mathcal{D}$ with $\operatorname{spt}\varphi \not\ni 0$, necessarily $u(\varphi) = 0$. Thus, by the theorem classifying such distributions, u is a linear combination of δ and its derivatives. Show that in fact $x\cdot u = 0$ implies that u is a multiple of δ itself.

[06.6] Given f in the Schwartz space $\mathscr{S}$, show that there is $F \in \mathscr{S}$ with $F' = f$ if and only if $\int_{\mathbb{R}} f = 0$.

[06.7] Let $u(x) = e^x\cdot\sin(e^x)$. Explain in what sense the integral $\displaystyle\int_{\mathbb{R}} f(x)\,u(x)\,dx$ converges for every $f \in \mathscr{S}$.

[06.8] Compute the Fourier transform of the sign function

$$\operatorname{sgn}(x) \;=\; \begin{cases} 1 & \text{(for } x>0\text{)} \\ -1 & \text{(for } x<0\text{)} \end{cases}$$

Hint: $\frac{d}{dx}\operatorname{sgn} \;=\; 2\delta$. Since Fourier transform converts d/dx to multiplication by $2\pi i x$, this implies that $(2\pi i)x\cdot\widehat{\operatorname{sgn}} = 2\widehat{\delta} = 2$. Thus, $(\pi i)x\cdot\widehat{\operatorname{sgn}} = 1$.

[06.9] On $\mathbb{R}^n$, show that $|x|^2\cdot\Delta\delta = 2n\cdot\delta$.

[06.10] On $\mathbb{R}^2$, compute the Fourier transform of $(x\pm iy)^n\cdot e^{-\pi(x^2+y^2)}$ for $n = 0,1,2,\ldots$. (*Hint:* Re-express things, including Fourier transform, in terms of $z = x+iy$ and $\overline{z} = x-iy$, $w = u+iv$, and $\overline{w} = u-iv$.)

[06.11] Show that on $\mathbb{R}^n$ with $n \geq 3$,

$$\Delta \frac{1}{|x|^{n-2}} \;=\; \text{constant multiple of } \delta$$

That is, up to a constant, $1/|x|^{n-2}$ is a *fundamental solution* for the Laplacian.

[06.12] In the context of complex analysis, the Cauchy-Riemann operator is

$$\frac{\partial}{\partial \overline{z}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

The Cauchy-Riemann equation characterizing holomorphic functions f is

$$\frac{\partial}{\partial \overline{z}} f \;=\; 0$$

Show that

$$\frac{\partial}{\partial \overline{z}} \frac{1}{z} \;=\; \text{constant multiple of } \delta$$

That is, $1/z$ is a *fundamental solution* for the Cauchy-Riemann operator. (So the shape of the Cauchy integral formula is perhaps not so surprising.)

[06.13] Show that, given a distribution u on $\mathbb{T}^n$, for any $0 \leq k \in \mathbb{Z}$ there is $f \in C^k(\mathbb{T}^n)$ and sufficiently large ℓ such that $(1-\Delta)^n f = u$.

[06.14] Show that, given a compactly-supported distribution u on $\mathbb{R}^n$, for any $0 \leq k \in \mathbb{Z}$ there is $f \in C^k(\mathbb{R}^n)$ and sufficiently large ℓ such that $(1-\Delta)^\ell f = u$.
