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Examples 07

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-disc-07.pdf]

For feedback on these examples, please get your write-ups to me by Friday, April 26, 2019.

[07.1] Recall the proof of the spectral theorem for self-adjoint operators on a finite-dimensional complex vector space V with hermitian inner product.

[07.2] Recall the proof of a spectral theorem for *two* self-adjoint operators S, T on V under the assumption that $ST = TS$.

[07.3] Let $K(x,y) = |x-y|$, and let

$$Tf(x) \;=\; \int_a^b K(x,y)\, f(y)\, dy \qquad\qquad (\text{for } f \in L^2[a,b])$$

Find some eigenvalues/eigenfunctions for the operator T. (*Hint*: consider $\frac{d^2}{dx^2}(Tf)$ and use the fundamental theorem of calculus.)

[07.4] Let $K(x,y) \in L^2([a,b]\times[a,b])$, and attempt to define a map $T : L^2[a,b] \to L^2[a,b]$ by

$$Tf(x) \;=\; \int_a^b K(x,y)\, f(y)\, dy$$

Show that Tf is well-defined a.e. as a pointwise-valued function. Show that T really does map L^2 to itself by showing that

$$|Tf|_{L^2[a,b]} \;\le\; |K|_{L^2([a,b]\times[a,b])} \cdot |f|_{L^2[a,b]}$$

(One would say that $K(,)$ is a *Schwartz kernel* for the map T. Yes, this use is in conflict with the use of *kernel* of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint T^* of T has Schwartz kernel $\overline{K(y,x)}$. In fact, the map T is a *Hilbert-Schmidt* operator, and is therefore *compact*.)

[07.5] Prove that the *Volterra operator* $Vf(x) = \displaystyle\int_0^x f(t)\, dt$ on $C^o[0,1]$ or on $L^2[0,1]$ has no (not-identically-zero) eigenvalues/eigenvectors (despite being *compact*!)

[07.6] Determine the spectrum of the *left-shift* $L : (c_1, c_2, \ldots) \to (c_2, \ldots)$ on ℓ^2, and of the *right-shift* $R : (c_1, c_2, \ldots) \to (0, c_1, c_2, \ldots)$ on ℓ^2. Show that these are mutual *adjoints*.

[07.7] *(Approximate eigenvectors and continuous spectrum, Weyl's criterion)* Let $T : V \to V$ be a self-adjoint linear operator on a Hilbert space V. For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) such that $(T-\lambda)v_n \to 0$ as $n \to +\infty$ is an *approximate eigenvector* for λ. Show that for λ *not* an eigenvalue for T, λ has an approximate eigenvector if and only if λ is in the spectrum of T.

[07.8] Show that the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Tf(x) = f(x)\cdot \sin x$ has empty discrete spectrum. Show that it is self-adjoint. Show that T has continuous spectrum the interval $[-1,1]$. (We know that self-adjoint (or merely *normal*) operators have only point spectum and continuous spectrum, that is, no left-over *residual* spectrum.)

[07.9] Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers inside the interval $[0,1]$. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the $r_1, r_2, \ldots$, and continuous spectrum the whole interval $[0,1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

[07.10] Let $r_1, r_2, r_3, \ldots$ be a bounded sequence of complex numbers. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is *compact* if and only if $r_n \to 0$.

[07.11] Let T be a compact operator $T : V \to W$ for Hilbert spaces V, W. For S a continuous/bounded operator on V, show that $T \circ S : V \to W$ is compact. For R a continuous/bounded operator on W, show that $R \circ T : V \to W$ is compact.

[07.12] Let S, T be two compact, self-adjoint operators on a Hilbert space, and $ST = TS$. Show that there is an orthonormal basis for V consisting of simultaneous eigenfunctions for S, T.

[07.13] Let $r_1, r_2, r_3, \ldots$ be a bounded sequence of complex numbers. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is *Hilbert-Schmidt* if and only if $\sum |r_n|^2 < \infty$.

[07.14] Let $r_1, r_2, r_3, \ldots$ be a bounded sequence of complex numbers. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is *trace class* if and only if $\sum |r_n| < \infty$.
