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Examples: discussion 01

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-01.pdf]

[01.1] Let $c_1 > c_2 > c_3 > \ldots > 0$ with $c_n \to 0$. Show that, for all $0 < x < 2\pi$, $\sum_n c_n e^{inx}$ is *convergent*.

Discussion: The expression as a Fourier series should not distract us from seeing an instance of the generalized alternating-decreasing criterion again, sometimes called *Dirichlet's criterion*: for a positive real sequence $c_1, c_2, \ldots$ monotone-decreasing to 0, and for a (possibly complex) sequence $b_1, b_2, \ldots$ with *bounded partial sums* $B_n = b_1 + \ldots + b_n$, the sum $\sum_n b_n c_n$ converges. The partial sums $\sum_{n\le N} e^{2\pi inx}$ are bounded for $0 < x < 1$, by summing *finite* geometric series:

$$\Big|\sum_{n=-M}^{N} z^n\Big| \;=\; \frac{|z^{-M} - z^{N+1}|}{|1-z|} \;\le\; \frac{2}{|1-z|}$$

so this criterion applies here.

The proof of the criterion itself is by *summation by parts*, a discrete analogue of integration by parts. That is, rewrite the tails of the sum as

$$\sum_{M\le n\le N} b_n c_n \;=\; \sum_{M\le n\le N} (B_n - B_{n-1})c_n \;=\; -B_{M-1}c_M + \sum_{M\le n\le N} B_n(c_n - c_{n+1}) + B_N c_{N+1}$$

Since the partial sums are bounded, the first and last summand go to 0. Letting β be a bound for all the $|B_n|$, the summation is

$$\Big|\sum_{M\le n\le N} B_n(c_n - c_{n+1})\Big| \;\le\; \sum_{M\le n\le N} |B_n|\cdot|c_n - c_{n+1}| \;=\; \sum_{M\le n\le N} |B_n|\cdot(c_n - c_{n+1}) \;\le\; \sum_{M\le n\le N} \beta\cdot(c_n - c_{n+1})$$

$$=\; \beta\cdot\sum_{M\le n\le N}(c_n - c_{n+1}) \;=\; \beta\cdot(c_M - c_{N+1})$$

by telescoping the series. Again, c_M and c_{N+1} go to 0. ///

[01.2] Let f_n be the piece-wise linear tent functions with heights n, widths $2/n$, centered at $1/n$: formulaically,

$$f_n(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0 \text{ or } x \ge \frac{2}{n}) \\ n^2\cdot x & (\text{for } 0 \le x \le \frac{1}{n}) \\ n - n^2(x - \frac{1}{n}) & (\text{for } \frac{1}{n} \le x \le \frac{2}{n}) \end{cases}$$

Show that for $g \in C^o(\mathbb{R})$,

$$\lim_n \int_{\mathbb{R}} f_n(x)\, g(x)\, dx \;=\; g(0)$$

Discussion: Certainly $f_n(x) = 0$ for $x \le 0$, for all n. For $x > 0$, there is positive integer n_o such that $x > \frac{2}{n_o}$, by the Archimedean property of the reals. Then for $n \ge n_o$ the functions f_n are 0 at x, giving the pointwise convergence to 0.

By design, the integrals of all the f_n are 1. Given $\varepsilon > 0$, let $\delta > 0$ be small enough so that $|g(x) - g(0)| < \varepsilon$ for $|x - 0| < \delta$. For n large enough so that $\frac{2}{n} < \delta$,

$$\Big|\int_{\mathbb{R}} f_n(x)\cdot g(x)\, dx \;-\; g(0)\Big| \;=\; \Big|\int_{\mathbb{R}} f_n(x)\cdot\big(g(x) - g(0)\big)\, dx\Big| \;=\; \int_0^{2/n} f_n(x)\cdot\big|g(x) - g(0)\big|\, dx$$

$$\leq \int_0^{2/n} f_n(x) \cdot \varepsilon \, dx \;=\; \varepsilon \cdot \int_0^{2/n} f_n(x) \, dx \;=\; \varepsilon$$

giving the convergence of the integrals to $g(0)$ as claimed. ///

[01.3] There is not much hope in making sense of the outcome of an uncountable number of non-zero operations: let Ω be an *uncountable* collection of positive real numbers. Letting F range over all finite subsets of Ω, show that $\sup_F \sum_{\alpha \in F} \alpha = +\infty$.

Discussion: Let $\Omega_1 = \{\omega \in \Omega : \omega > 1\}$, and for $n = 2, 3, \ldots$, let $\Omega_n = \{\omega \in \Omega : \frac{1}{n} < \omega \leq \frac{1}{n-1}\}$. There are countably many such sets, so in (at least) one of them Ω_{n_o} there must be infinitely-many elements of Ω (or else Ω would be a countable union of countable sets, hence countable). Then

$$\sup_F \sum_{\alpha \in F} \;\geq\; \sup_{F \subset \Omega_{n_o}} \sum_{\alpha \in F} \;\geq\; \sup_{F \subset \Omega_{n_o}} \#F \cdot \frac{1}{n_o} \;=\; \frac{1}{n_o} \sup_{F \subset \Omega_{n_o}} \#F \;=\; +\infty$$

because Ω_{n_o} is infinite. ///

[01.4] Show that the closed unit ball in ℓ^2 is *not compact*.

Discussion: Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with the single 1 at the n^{th} place. Then $d(e_m, e_n) = \sqrt{2}$ for $m \neq n$. Thus, the sequence of e_n's has no Cauchy subsequence, so no convergent subsequence. ///

[01.5] Show that the closed unit ball in $C^o[a, b]$ is not compact.

Discussion: For example, let f_n be a tent function centered at $1/2^n$, of height 1, and width $1/2^{n+2}$ (or anything strictly larger than $1/2^{n+1}$). By design, the supports of these functions are disjoint, and all their sup-norms are 1. Thus, for $m \neq n$, $|f_m - f_n|_{C^o} = 1$. Thus, the sequence has no Cauchy subsequence. ///

[01.6] Show that $C^o[a, b]$ is *not complete* with the $L^2[a, b]$ metric.

Discussion: That is, we want a sequence $\{f_n\}$ of C^o functions that is Cauchy in the L^2 metric, but not in the C^o metric. In particular, it would suffice to find $\{f_n\}$ which converge in L^2 to an L^2 function which is not C^o.

For example, $\{f_n\}$ can be a sequence of continuous, piecewise-linear functions converging pointwise to a step function (which is certainly not continuous). For example, with $[a, b] = [0, 1]$,

$$f_n(x) \;=\; \begin{cases} 0 & (\text{for } 0 \leq x < \frac{1}{2} - \frac{1}{n}) \\ \frac{n}{2} \cdot (x - \frac{1}{2} + \frac{1}{n}) & (\text{for } \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n}) \\ 1 & (\text{for } \frac{1}{2} + \frac{1}{n} < x \leq 1) \end{cases}$$

The graph is flat to the left and flat to the right, and has a straight line of slope $n/2$ connecting the two flat parts. The pointwise limit is a step function with step of height 1 at $\frac{1}{2}$.

For $m \leq n$ the L^2 norm of $f_m - f_n$ is easily estimated by

$$|f_m - f_n|^2_{L^2} \;=\; \int_{\frac{1}{2} - \frac{1}{m}}^{\frac{1}{2} + \frac{1}{m}} |f_m(x) - f_n(x)|^2 \, dx \;\leq\; \int_{\frac{1}{2} - \frac{1}{m}}^{\frac{1}{2} + \frac{1}{m}} 1 \, dx \;\leq\; \frac{2}{m}$$

Thus, the sequence is L^2-Cauchy. Since the limit is not continuous, we imagine that the sequence cannot possibly be C^o-Cauchy. But that is not quite a proof.

Explicitly, $|f_n - f_{2n}|_{C_o} = \frac{1}{4}$. So the sequence is definitely not C^o-Cauchy, while it is nevertheless L^2-Cauchy. Still, this does not immediately entail that adjusting the step function on a set of measure zero (getting a little ahead of ourselves) could not make it continuous. Or, more elementarily, in fact we have not proven that the apparent L^2-norm really is a norm on various pointwise limits of sequences of continuous functions. Indeed, it is not! For example, narrowing tents can give a pointwise limit

$$g(x) \;=\; \begin{cases} 0 & \text{for } x \in [0,1] \text{ but } x \neq \frac{1}{2} \\ \\ 1 & \text{for } x = \frac{1}{2} \end{cases}$$

We can Riemann-integrate such functions, and $|g|_{L^2} = 0$, yet g is not the pointwise-everywhere-zero function, so there is a problem with the norm.

But we don't quite need that strong (false) assertion.

In this particular example, let s be the step function, and f an alleged continuous function such that $|s - f|_{L^2[0,1]} = 0$. We use the fact that s *is* continuous on both $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$, and

$$|s-f|^2_{L^2[0,1]} \;=\; |s-f|^2_{L^2[0,\frac{1}{2})} + |s-f|^2_{L^2(\frac{1}{2},1]}$$

In particular,

$$|s-f|^2_{L^2[0,\frac{1}{2})} \;=\; 0 \;=\; |s-f|^2_{L^2(\frac{1}{2},1]}$$

If the continuous function $s - f$ on $[0, \frac{1}{2})$ is non-zero anywhere on that interval, then by continuity there is $\varepsilon > 0$ and $h > 0$ such that $|s(x) - f(x)| > \varepsilon$ on a subinterval of length at least h. Thus, we would have $|s-f|^2_{L^2[0,\frac{1}{2}]} \geq \varepsilon^2 \cdot h$, which is impossible. The same argument applies to $s - f$ on $(\frac{1}{2}, 1]$.

Thus, we have found that $s(x) = f(x)$ for $x \in [0, \frac{1}{2})$, and also for $x \in (\frac{1}{2}, 1]$. But this is impossible for continuous f on $[0, 1]$. Thus, there is *no* continuous f such that $|s - f|_{L^2[0,1]} = 0$. ///

[0.1] Remark: The previous argument would also apply with the sign function replaced by piecewise continuous functions, but more general possibilities are unclear.

[01.7] Show that $C^1[a, b]$ is not complete with the $C^o[a, b]$ metric.

Discussion: One approach is to find a C^o-Cauchy sequence of C^1 functions whose limit is not C^1. For example, in words, a tent function with base $[-1, 1]$ with vertex at 0 is continuous, but not differentiable. It can be approximated in C^o by tent functions that are smoothed off in tinier-and-tinier intervals around the vertex.

(In contrast to the case of L^2-not-completeness of C^o, here pointwise values of functions are well-defined and correctly determine their properties.)

If we demand explicit formulas rather than pictures, it's a question of writing formulas for (for example) little pieces of pointier-and-pointier parabola pieces to replace the sharp corner at the peak of the tent function. For example, anticipating that $f_n(x)$ is $1 - |x|$ for $|x| \geq 1/n$, to match values and slopes at $\pm 1/n$ with a parabola $1 - a_n - b_n x^2$ (with that $1 - a_n$ suggested by the picture), we need

$$1 - a_n - b_n \cdot \frac{1}{n^2} \;=\; 1 - \frac{1}{n} \qquad \text{and} \qquad -2b_n \cdot \frac{1}{\pm n} \;=\; \mp 1$$

Thus,

$$f_n(x) \;=\; 1 - \frac{1}{2n} - \frac{n}{2}x^2 \qquad (\text{for } |x| \leq 1/n)$$

smooths the corner, for every n.

Losing interest in this approach... Is there a better one? Non-formulaic? Seriously, turning obvious pictures into formulas quickly becomes unrewarding and non-explanatory. Yes: we should soon prove that $C^\infty[a,b]$ is dense in all the spaces $C^k[a,b]$, without writing dubious formulas. This changes the presentation of the question, but annihilates it. ///

[01.8] Show that $C^1[a,b]$ is complete, with the $C^1[a,b]$ metric

$$d(f,g) \;=\; \sup_{a\le x\le b} |f(x)-g(x)| + \sup_{a\le x\le b} |f'(x)-g'(x)|$$

Discussion: For a Cauchy sequence $\{f_i\}$ in $C^k[a,b]$, the pointwise limits $\lim_i f(x)$ and $\lim_i f'(x)$ exist, and are continuous, since the limits are uniform pointwise. The issue is to show that $\lim_i f$ is differentiable, with derivative $\lim_i f'$. That is, for a Cauchy sequence f_n in $C^1[a,b]$, with pointwise limits $f(x) = \lim_n f_n(x)$ and $g(x) = \lim_n f_n'(x)$, we have $g = f'$. By the fundamental theorem of calculus, for any index i,

$$f_i(x) - f_i(a) \;=\; \int_a^x f_i'(t)\,dt$$

Since the f_i' *uniformly* approach g, given $\varepsilon > 0$ there is i_o such that $|f_i'(t) - g(t)| < \varepsilon$ for $i \ge i_o$ and for *all* t in the interval, so for such i

$$\Big|\int_a^x f_i'(t)\,dt - \int_a^x g(t)\,dt\Big| \;\le\; \int_a^x |f_i'(t) - g(t)|\;dt \;\le\; \varepsilon\cdot|x-a| \;\longrightarrow\; 0$$

Thus,

$$\lim_i f_i(x) - f_i(a) \;=\; \lim_i \int_a^x f_i'(t)\,dt \;=\; \int_a^x g(t)\,dt$$

from which $f' = g$. ///

[01.9] Show that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals r_n,

$$C(r) \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le r_n\}$$

is compact if and only if $\sum_n |r_n|^2 < \infty$.

Discussion: Use the *total boundedness* criterion. Given $\varepsilon > 0$, by convergence of $\sum_n \delta_n^2$, there is n_o large enough so that $\sum_{n\ge n_o} \delta_n^2 < \varepsilon^2$. The set

$$C_{n_o} \;=\; \{(z_1, z_2, \ldots, z_{n_o}) \in \mathbb{R}^{n_o} : |z_n| \le \delta_n\}$$

is a compact subset of $\mathbb{R}^{n_o}$, so certainly has a finite cover by open balls of radius ε. Let the centers of these balls be $w_1, \ldots, w_N$. Let $j : \mathbb{R}^{n_o} \to \ell^2$ be the inclusion $j(z_1, \ldots, z_{n_o}) = (z_1, \ldots, z_{n_o}, 0, 0, \ldots)$. Then we claim that the open balls of radius 2ε at $j(w_1), j(w_2), \ldots, j(w_N)$ cover $C(\delta)$. Indeed, given $z = (z_1, z_2, \ldots) \in C(\delta)$, write $z = j(z') + z''$ where $z' = (z_1, \ldots, z_{n_o})$ and $z'' = z - j(z') = (0, \ldots, 0, z_{n_o+1}, \ldots)$. There is at least one of the w_js within ε of z': let w_{j_o} be such. By the triangle inequality for the norm $|\cdot|_{\ell^2}$ on ℓ^2,

$$d(z, j(w_{j_o})) \;=\; |z - j(w_{j_o})|_{\ell^2} \;=\; |j(z') + z'' - j(w_{j_o})|_{\ell^2} \;\le\; |j(z') - j(w_{j_o})|_{\ell^2} + |z''|_{\ell^2}$$

$$=\; |z' - w_{j_o}|_{\mathbb{R}^{n_o}} + |z''|_{\ell^2} \;<\; \varepsilon + \varepsilon$$

Thus, $C(r)$ can be covered by finitely-many open balls of radius 2ε.

Conversely, if $\sum_n r_n^2 = +\infty$, then there are indices $1 \le n_1 < n_2 < \ldots$ such that

$$\sum_{n_k < i \le n_{k+1}} r_n^2 \;\ge\; 1$$

With standard basis $\{e_n\}$, let

$$v_k \;=\; \sum_{n_k < i \le n_{k+1}} r_i \cdot e_i$$

Then for $k \neq \ell$,

$$|v_k - v_\ell|^2 \;=\; \sum_{n_k < i \le n_{k+1}} r_i^2 + \sum_{n_\ell < i \le n_{\ell+1}} r_i^2 \;\ge\; 1+1$$

Thus, there are no convergent subsequences, and $C(r)$ is not sequentially compact, so not compact. ///

[01.10] The space of continuous functions on $\mathbb{R}$ *going to* 0 *at infinity* is

$$C_o^o(\mathbb{R}) \;=\; \{f \in C^o(\mathbb{R}) : \text{for every } \varepsilon > 0 \text{ there is } T \text{ such that } |f(x)| < \varepsilon \text{ for all } |x| \ge T\}$$

Show that the closure of $C_c^o(R)$ in the space $C^o_{\text{bdd}}(\mathbb{R})$ of bounded continuous functions with sup norm, is $C_o^o(\mathbb{R})$.

Discussion: The argument for this is general enough that we can replace $\mathbb{R}$ by a more general topological space X, probably locally compact and Hausdorff so that Urysohn's lemma assures us a good supply of continuous functions for auxiliary purposes. Then $C_o^o(X)$ is defined to be the collection of continuous functions f such that, given $\varepsilon > 0$, there is a compact $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$.

First, show that any $f \in C_o^o(\mathbb{R})$ is a sup-norm limit of functions from $C_c^o(\mathbb{R})$. Given $\varepsilon > 0$, let K be sufficiently large so that $|f(x)| < \varepsilon$ for $x \notin K$. We claim that there is an open $U \supset K$ with compact closure $\overline{U}$ (which would be obvious on $\mathbb{R}$ or $\mathbb{R}^n$). For each $x \in K$, let $U_x \ni x$ be an open set with compact closure (using the local compactness). By compactness of K, there is a finite subcover $K \subset U_{x_1} \cup \ldots \cup U_{x_n}$. Then the closure of $U = U_{x_1} \cup \ldots \cup U_{x_n}$ is compact, as claimed. Then, invoking Urysohn's Lemma, let φ be a continuous function on X taking values in the interval $[0,1]$, that is 1 on K, and 0 off U, so φ has compact support. Then $\varphi \cdot f$ is continuous and has compact support, and

$$\sup_{x \in X} |f(x) - \varphi(x) \cdot f(x)| \;\le\; \sup_{x \in K} |f(x) - \varphi(x) \cdot f(x)| + \sup_{x \notin K} |f(x) - \varphi(x) \cdot f(x)| \;=\; 0 + \sup_{x \notin K} |f(x) - \varphi(x) \cdot f(x)|$$

$$\le\; \sup |1 - \varphi| \cdot \sup_{x \notin K} |f(x)| \;<\; 1 \cdot \varepsilon$$

That is, we can approximate f to within ε, as claimed.

On the other hand, now show that any sup-norm Cauchy sequence of $f_n \in C_c^o(X)$ has a pointwise limit f in $C_o^o(X)$. First, on any compact, the limit of the f_n's is *uniform* pointwise, so is continuous on compacts. Since every point $x \in X$ has a neighborhood U_x with compact closure, the pointwise limit is continuous on U_x. Thus, the pointwise limit is continuous at every point, hence continuous. Given $\varepsilon > 0$, take n_o sufficiently large so that $\sup_{x \in X} |f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \ge n_o$. Let K be the support of f_{n_o}. Then

$$\sup_{x \notin K} |f(x)| \;=\; \sup_{x \notin K} |f(x) - f_{n_o}(x)| \;\le\; \sup_{x \in X} |f(x) - f_{n_o}| \;\le\; \varepsilon$$

Thus, the pointwise limit goes to 0 at infinity. ///
