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Examples: discussion 02

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-02.pdf]

[02.1] Show that every open subset of $\mathbb{R}^n$ is a *countable* union of open balls.

Discussion: We can prove a stronger result, that in a metric space with a countable dense subset D, every open is a countable union of open balls. Of course, $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$. Let U be the open set. For $x \in U$, let $B(r_x, x)$ be an open ball or radius r_x centered at x and contained in U. We can shrink r_x to make it rational. By density, there is an element $d_x \in D$ in the smaller ball $B(r_x/2, x)$. Then $B(r_x/2, d_x)$ contains x and is inside $B(r_x, x)$, so is inside U. Thus, $U \subset \bigcup_{x\in U} B(r_x/2, d_x)$. By countability of D and of rationals (the radii), there can be only countably-many distinct balls $B(r_x, d_x)$. ///

[02.2] For positive real $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\le\; w_1 a_1 + \ldots + w_n a_n$$

Discussion: This is a corollary of Jensen's inequality, similar to the arithmetic-geometric mean, but with unequal weights. Namely, let $X = \{1, 2, \ldots, n\}$ with measure $\mu(i) = w_i$, and function $g(i) = \log a_i$. Then Jensen's inequality (with $f(x) = e^x$) is

$$\exp\Big(\sum_{i=1}^n w_i \cdot \log a_i\Big) \;\le\; \sum_{i=1}^n w_i \cdot e^{\log a_i}$$

which simplifies to the assertion. ///

[02.3] *Lebesgue (outer) measure* $\mu(E)$ of subsets E of $\mathbb{R}$ is

$$\mu(E) \;=\; \inf\{\sum_{n=1}^\infty |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^\infty (a_n, b_n)\}$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where M is Cantor's middle-thirds set.

Discussion: Enumerate the rationals as $r_1, r_2, \ldots$. Given $\varepsilon > 0$, let $U_{n,\varepsilon}$ be the interval $(r_n - \frac{\varepsilon}{2^n}, r + \frac{\varepsilon}{2^n})$. The union of these intervals contains $\mathbb{Q}$, and the sum of lengths is $\varepsilon \cdot (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = \varepsilon$.

The Cantor middle-thirds set can be described in terms of base-three expansions, as follows. All real numbers r in $[0, 1]$ have (ternary) expansion $r = \sum_{n=1}^\infty \frac{a_n}{3^n}$ with all coefficients a_n in the set $\{0, 1, 2\}$. The expansion is unambiguous except for the possibility of coefficients all 2 beyond a certain point, which we exclude by using

$$\frac{2}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \ldots \;=\; 2 \cdot \frac{3^{-n}}{1 - \frac{1}{3}} \;=\; 2 \cdot \frac{3^{1-n}}{3 - 1} \;=\; 3^{1-n}$$

Then the middle-thirds set C is the set of reals $r = \sum_{n=1}^\infty \frac{a_n}{3^n}$ with all coefficients a_n in the set $\{0, 2\}$ (with the convention excluding endlessly repeating 2's).

Alternatively, the middle-thirds set C is formed as a *nested intersection*, as follows. Let C_1 be $[0, 1]$ with the middle third $(\frac{1}{3}, \frac{2}{3})$ removed. Let C_2 be C_1 with the middle third thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ removed, and so on. At each step, the sum of lengths of the remaining intervals is multiplied by $(1 - \frac{1}{3}) = \frac{2}{3}$, and the number of intervals is multiplied by 2. After n middle-third removals, the result C_n is a union of 2^n intervals each of length 3^{-n}. The Cantor middle-thirds set is $C = \bigcap_n C_n$.

Given $\varepsilon > 0$, choose n large enough so that $2^n/3^n < \varepsilon/2$. Cover each of the 2^n intervals of length 3^{-n} making up C_n by an open interval of length $2 \cdot 3^{-n}$. The sum of the lengths of these 2^n open intervals is

$$2^n \cdot (2 \cdot 3^{-n}) \;=\; 2 \cdot (2/3)^n \;<\; 2 \cdot \frac{\varepsilon}{2} \;=\; \varepsilon$$

This exhibits an open cover of C_n with sum of lengths less than ε. Since $C \subset C_n$, this gives such a cover of C itself, as desired. ///

[02.4] Show that for measurable f on $[a,b]$,

$$\Big| \int_a^b f(x)\, dx \Big|^2 \;\le\; |b-a| \cdot \int_a^b |f(x)|^2\, dx$$

with equality only for f (almost-everywhere) constant.

Discussion: This is an instance of an *extended* version of Cauchy-Schwarz-Bunyakowsky, in which $+\infty$ is allowed as a value of either side, as in Hölder's inequality. Thus, if the right-hand side is finite, $f \in L^2[a,b]$, and this the ordinary Cauchy-Schwarz-Bunyakowsky, with $|\langle f, 1\rangle|^2 \le |1|^2_{L^2[a,b]} \cdot |f|^2_{L^2[a,b]}$. If the right-hand side is $+\infty$, then the inequality is vacuously true. ///

[02.5] For non-negative, real-valued f, show that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x)\, dx$$

Discussion: We should imagine that this question is to be an application of Lebesgue's Monotone Convergence, but it is posed in a sloppy way, since Monotone Convergence refers to *sequences*. So we reframe the question: given a monotone decreasing sequence $\varepsilon_n \to 0^+$, show that

$$\lim_n \int_{\mathbb{R}} f(x)\, e^{-\varepsilon_n x^2}\, dx \;=\; \int_{\mathbb{R}} f(x)\, dx$$

Then Monotone Convergence applies, since

$$f(x)\, e^{-\varepsilon_n x^2} \;\le\; f(x)\, e^{-\varepsilon_{n+1} x^2} \qquad \text{(for all } x \in \mathbb{R} \text{ and for all } n\text{)}$$

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[02.6] For $g \in C^o_o(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, show that

$$\lim_{t \to +\infty} \int_{\mathbb{R}} f(x)\, g(x+t)\, dx \;=\; 0$$

Discussion: We should imagine that this question is to be an application of Lebesgue's Dominated Convergence, but it is posed in a sloppy way, since Dominated Convergence refers to *sequences*. So we reframe the question: given a sequence $t_n \to +\infty$, show that

$$\lim_n \int_{\mathbb{R}} f(x)\, g(x+t_n)\, dx \;=\; 0 \;=\; \int_{\mathbb{R}} f(x)\, dx$$

Since $g \in C^o_o$, $|g|$ has a finite bound B, and

$$|f(x) \cdot g(x+t_n)| \;\le\; |f(x) \cdot B| \;\in\; L^1(\mathbb{R}) \qquad \text{(for all } n\text{, for all } x \in \mathbb{R}\text{)}$$

Thus, Dominated Convergence applies. Without loss of generality, we can suppose that $|f|$ takes values in $\mathbb{R}$ everywhere, as opposed to $+\infty$, by adjusting f if necessary on a set of measure 0. Since $g \in C_o^o$, given $\varepsilon > 0$, there is T such that $|g(x)| < \varepsilon$ for $|x| \geq T$.

Fix $x \in \mathbb{R}$. Take n_o large enough so that $|x+t_n| \geq T$ for all $n \geq n_o$. Then $|f(x)g(x+t_n)| < \varepsilon \cdot |f(x)|$. Thus, $|f(x)g(x+t_n)| \to 0$. That is, the functions $x \to F_n(x) = |f(x)g(x+t_n)|$ go to 0 pointwise. By Dominated Convergence, the limit of the integrals is the integral of the limit, 0, so is 0. ///

[02.7] Functions in $L^1(\mathbb{R})$ need not go to 0 at infinity: give an example of $f \in L^1(\mathbb{R})$ such that $\limsup_{x\to+\infty} |f(x)| = +\infty$.

Discussion: Let f_n be a piecewise linear tent of height n and width 2^{-n} centered at n. Then $\int f_n = n/2^n$, and $f = \sum_n f_n$ is in $L^1(\mathbb{R})$, but certainly $\limsup_{x\to+\infty} |f(x)| = +\infty$. ///

[02.8] For $f \in L^1(\mathbb{R})$, show that

$$\lim_{\varepsilon\to 0^+} \int_0^\varepsilon f(x)\, dx \;=\; 0$$

Discussion: Let $S_n = \{x : \frac{1}{n+1} \leq |x| < \frac{1}{n}\}$. Then

$$\Big|\sum_{n\geq 1} \int_{S_n} f\Big| \;\leq\; \sum_{n\geq 1} \int_{S_n} |f| \;\leq\; |f|_{L^1}$$

Thus, the sum of non-negative terms $\sum_{n\geq 1} \int_{S_n} |f|$ is convergent, so the tails $\sum_{n\geq N} \int_{S_n} |f|$ go to 0 as $N \to +\infty$. Thus,

$$\Big|\int_{|x|\leq 1/N} f\Big| \;\leq\; \int_{|x|\leq 1/N} |f| \;=\; \sum_{n\geq N} \int_{S_n} |f|$$

goes to 0 as $N \to +\infty$. Then this idea can be applied to $\int_{|x|<\delta} |f|^p$ in the previous example. ///

[02.9] For $f \in L^2(\mathbb{R})$, show that there is a constant C such that

$$\Big|\int_0^\varepsilon f(x)\, dx\Big| \;\leq\; C \cdot \sqrt{\varepsilon}$$

for $0 < \varepsilon \leq 1$.

Discussion: Let h_ε be the characteristic function of $[0, \varepsilon]$. By Cauchy-Schwarz-Bunyakowsky

$$\Big|\int_0^\varepsilon f\Big| \;=\; |\langle f, h_\varepsilon\rangle_{L^2(\mathbb{R})}| \;\leq\; |f|_{L^2(\mathbb{R})} \cdot |h_\varepsilon|_{L^2(\mathbb{R})} \;=\; |f|_{L^2(\mathbb{R})} \cdot \sqrt{\varepsilon}$$

The case of conjugate exponents $\frac{1}{p} + \frac{1}{q} = 1$ is the same, using Hölder's inequality rather than Cauchy-Schwarz-Bunyakowsky. There is no immediate analogue for L^1, although a weaker result is possible, as in the previous example. ///

[02.10] Let f be a continuous function on $[0,1]$, with $f(0) = 0$ and $f(1) = 1$. Show that the set $\{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\}$ has positive measure.

Discussion: By continuity $E = f^{-1}(1/4, 3/4)$ is open. By the intermediate value theorem, $E \neq \phi$. Let $x_o \in E$. Since E is open, there is some open interval $(x-\delta, x+\delta) \subset E$, for $\delta > 0$. Then the measure of E is at least $2\delta > 0$. ///

[02.11] Show that $\ell^p \subset \ell^q$ for $1 < p < q < \infty$, and that the containment is *proper*.

Discussion: It is worthwhile to carry out a fuller version of this exercise. Take $p < p'$. We claim that $L^p[a,b] \supset L^{p'}[a,b]$, with proper containment. The function f that is $(x-a)^{-\frac{1}{p'}}$ on $(a,b]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p. Given $f \in L^{p'}[a,b]$, let E be the set of $x \in [a,b]$ where $|f(x)| \geq 1$. Then $\int_a^b |f|^{p'} < \infty$ if and only if $\int_E |f|^{p'} < \infty$. On E, $|f|^p < |f|^{p'}$, so $\int_E |f|^p < \infty$, and then also $\int_a^b |f|^p < \infty$, so $f \in L^p[a,b]$. ///

We claim that $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$ are not comparable for $p \neq p'$. Take $1 \leq p < p'$. On one hand, $1/(1+|x|)^{1/p'+\varepsilon}$ is in $L^{p'}$ for all $\varepsilon > 0$, but not in L^p for ε small enough so that $\frac{1}{p'} + \varepsilon < \frac{1}{p}$. On the other hand, the function f that is $x^{-\frac{1}{p'}}$ on $(0,1]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p.

We claim that for $1 \leq p < p' < \infty$, $\ell^p \subset \ell^{p'}$, with strict containment. Indeed, $f(n) = \frac{1}{n^{1/p}}$ is not in ℓ^p, but is in $\ell^{p'}$. Let $E = \{n \in \{1,2,\ldots\} : |f(n)| < 1\}$. Then $f \in \ell^p$ if and only if the *complement* of E is finite, and if $\sum_{n \in E} |f(n)|^p < \infty$. Certainly $|f(n)|^p > |f(n)|^{p'}$ for $n \in E$, and the complement of E is finite, so $\sum_{n \in E} |f(n)|^{p'} < \sum_{n \in E} |f(n)|^p$, and $f \in \ell^{p'}$. ///
