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Examples: discussion 03

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[03.1] For a vector subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the topological closure of W.

Discussion: Let $\lambda_x(v) = \langle v, x\rangle$ for $x, v \in V$. Then $W^\perp = \bigcap_{w\in W} \ker \lambda_w$. Similarly, $(W^\perp)^\perp = \bigcap_{x\in W^\perp} \ker \lambda_x$. From the discussion in the Riesz-Fréchet theorem, or directly via Cauchy-Schwarz-Bunyakowsky, each λ_x is continuous, so $\ker \lambda_x = \lambda_x^{-1}(\{0\})$ is closed, since $\{0\}$ is closed. (One might check that the kernel of a linear map is a vector subspace.) An arbitrary intersection of closed sets is closed, so $(W^\perp)^\perp$ is closed.

Certainly $(W^\perp)^\perp \supset W$, because for each $w \in W$, $\langle x, w\rangle = 0$ for all $x \in W^\perp$. Thus, $(W^\perp)^\perp$ is a closed subspace, containing W. Being a closed subspace of a Hilbert space, $(W^\perp)^\perp$ is a Hilbert space itself. If $(W^\perp)^\perp$ were strictly larger than the topological closure $\overline{W}$ of W, then there would be $0 \neq y \in (W^\perp)^\perp$ orthogonal to $\overline{W}$. Then y would be orthogonal to W itself, so $0 \neq y \in W^\perp$, contradicting $0 \neq y \in (W^\perp)^\perp$. ///

[03.2] Find two *dense* vector subspaces X, Y of ℓ^2 such that $X \cap Y = \{0\}$. (And, if you need further entertainment, can you find countably-many dense vector subspaces X_n such that $X_m \cap X_n = \{0\}$ for $m \neq n$?)

Discussion: First, as a variant that refers to more natural constructions, but requires non-trivial proofs to fully validate it, we can make two dense subspaces of $L^2[0,1]$ which intersect just at $\{0\}$. Namely, the vector space of all finite Fourier series, and the vector space of all polynomials (restricted to $[0,1]$). We need to know that the appropriate exponentials (or sines and cosines) give a Hilbert space basis of $L^2[0,1]$, and also Weierstraß' result on the density of polynomials in $C^o[0,1]$, hence (depending on our definitional set-up) in $L^2[0,1]$.

A more elementary, but trickier, approach is the following. Let X be the vector space of *finite* linear combinations of the standard Hilbert space basis $\{e_n\}$. This is a natural subspace. For the other subspace Y, *some* sort of trickery seems to be needed, either in specification of Y itself so as to make verification of $X \cap Y = \{0\}$ easy, or a simpler specification of Y but with complicated verification that $X \cap Y = \{0\}$, or both.

One possibility involves Sun-Ze's theorem (sometimes called the Chinese Remainder Theorem), namely, that for a finite collection of mutually relatively prime integers $N_1, \ldots, N_k$, and for integers $b_1, \ldots, b_k$ there exists $x \in \mathbb{Z}$ such that $x = b_k \bmod N_k$. Further, this x can be arbitrarily large, by adding multiples of the product $N_1 \ldots N_k$ to it. Let p_n be the n^{th} prime number, and put

$$v_n \;=\; e_n + \sum_{k\geq 1} \frac{1}{kp_n} \cdot e_{kp_n}$$

Of course, we claim that no (non-zero) finite linear combination $y = \sum_n c_n \cdot v_n$ is in X. That is, we claim that for any such non-zero linear combination, there are arbitrarily large indices ℓ such that $\langle y, e_\ell\rangle \neq 0$. Let n_o be the largest index n such that $c_n \neq 0$. Invoking Sun-Ze's theorem, there exist $\ell \geq n_o$ such that $\ell = 1 \bmod p_i$ for $i < n_o$ and $\ell = 0 \bmod p_{n_o}$. Then

$$\langle y, e_\ell\rangle \;=\; \sum_n \Big(\frac{1}{n}\langle e_n, e_\ell\rangle + \sum_k \frac{1}{kp_n}\langle e_{kp_n}, e_\ell\rangle\Big) \;=\; \sum_{n<n_o} 0 \;+\; \frac{1}{\ell} \;\neq\; 0$$

This proves that $X \cap Y = \{0\}$.

Certainly X is dense, because every vector in ℓ^2 is an infinite sum of vectors from X, that is, an ℓ^2 limit of finite linear combinations of vectors from X.

To see that Y is dense, observe that applying an *infinite* version of Gram-Schmidt to the vectors v_n produces the standard basis e_n. That is, the e_n's are *infinite* linear combinations of the v_n's, so Y is dense. (Yes, there is an issue about *convergence* in an infinite version of Gram-Schmidt, in general!) ///

[03.3] For measurable $E \subset [0,1]$, show that $\lim_n \int_E e^{-2\pi inx}\,dx = 0$ as $n \to \infty$ ranging over integers.

Discussion The characteristic function χ_E is in $L^2[0,1]$, and

$$\int_E e^{-2\pi inx}\,dx \;=\; \int_0^1 e^{-2\pi inx}\chi_E(x)\,dx$$

which goes to 0, by the abstract/easy Riemann-Lebesgue Lemma. ///

[03.4] Let $f_n(x) = \sin \pi nx$ on $[0,1]$, extended by $\mathbb{Z}$-periodicity, for $n = 1,2,3,\ldots$. Given $g \in L^1[0,1]$, show that $\int_0^1 f_n \cdot g \to 0$.

Discussion If in fact $g \in L^2[0,1]$, then the more elementary/abstract Riemann-Lebesgue Lemma immediately gives the assertion, because the functions f_n are orthonormal. (We do not need *completeness* to reach this conclusion.)

But $L^1[0,1]$ is not contained in $L^2[0,1]$, because of functions like $g(x) = 1/\sqrt{x}$. Nevertheless, we can *extend* g by 0 to a function $h \in L^1(\mathbb{R})$. Then

$$\int_0^1 f_n \cdot g \;=\; \int_{\mathbb{R}} \sin \pi nx \cdot h(x)\,dx$$

which goes to 0 by the more substantive, less abstract, Riemann-Lebesgue Lemma. ///

[03.5] Compute the Fourier coefficients of the sawtooth function $s(x) = x - \frac{1}{2}$ on $[0,1]$, extended by $\mathbb{Z}$-periodicity. Use this to show that $\sum_{n\ge 1} 1/n^2 = \pi^2/6$.

Discussion: We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0,2\pi]$. The Fourier coefficients are determined by Fourier's formula

$$\widehat{f}(n) \;=\; \int_0^{2\pi} f(x)\,\frac{e^{-inx}}{\sqrt{2\pi}}\,dx$$

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get

$$\widehat{f}(n) \;=\; \Big[f(x)\cdot \frac{e^{-inx}}{\sqrt{2\pi}\cdot(-in)}\Big]_0^{2\pi} - \int_0^{2\pi} 1 \cdot \frac{e^{-inx}}{\sqrt{2\pi}\cdot(-in)}\,dx$$

$$=\; \Big((\pi \cdot \frac{1}{\sqrt{2\pi}\cdot(-in)}) - (-\pi \cdot \frac{1}{\sqrt{2\pi}\cdot(-in)})\Big) - 0 \;=\; \frac{2\pi}{\sqrt{2\pi}\cdot(-in)} \;=\; \frac{\sqrt{2\pi}}{-in}$$

The L^2 norm of f is

$$\int_0^{2\pi} (x-\pi)^2\,dx \;=\; \Big[\frac{(x-\pi)^3}{3}\Big]_0^{2\pi} \;=\; \frac{\pi^3 - (-\pi)^3}{3} \;=\; \frac{2\pi^3}{3}$$

Thus, by Parseval,

$$\sum_{n\neq 0} \Big|\frac{\sqrt{2\pi}}{-in}\Big|^2 \;=\; \frac{2\pi^3}{3}$$

This simplifies first to

$$2\sum_{n\geq 1}\frac{2\pi}{n^2} \;=\; \frac{2\pi^3}{3}$$

and then to

$$\sum_{n\geq 1}\frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

That is, Parseval applied to the sawtooth function evaluates $\zeta(2)$. ///

[03.6] Let E be a Lebesgue measurable set in $\mathbb{R}$ with finite Lebesgue measure. Show that

$$\lim_{t\to+\infty}\int_E \sin tx\;dx \;=\; 0 \qquad\qquad \text{(over real } t\text{)}$$

Discussion The characteristic function χ_E of E is in $L^1(\mathbb{R})$ since E has finite measure, and

$$\int_E \sin tx\;dx \;=\; \int_{\mathbb{R}} \chi_E(x)\cdot \sin tx\;dx$$

which goes to 0 by the Riemann-Lebesgue Lemma. ///

[03.7] Compute $\int_{\mathbb{R}}\left(\frac{\sin x}{x}\right)^2\,dx$. (*Hint:* do not attempt to do this directly, nor by complex analysis.)

Discussion: From a standard stock of easy Fourier transforms, the Fourier transform of a characteristic function of a symmetrical interval is very close to the given function:

$$\widehat{\mathrm{ch}_{[-1,1]}}(\xi) \;=\; \int_{-1}^{1} e^{-2\pi i\xi x}\;dx \;=\; \frac{e^{-2\pi i\xi}-e^{2\pi i\xi}}{-2\pi i\xi} \;=\; \frac{\sin 2\pi\xi}{\pi\xi}$$

Applying Plancherel, we have

$$2 \;=\; \int_{\mathbb{R}} |\mathrm{ch}_{[-1,1]}|^2 \;=\; \int_{\mathbb{R}}\left(\frac{\sin 2\pi\xi}{\pi\xi}\right)^2 d\xi$$

The change of variables replacing ξ by $\xi/2\pi$ gives

$$2 \;=\; \int_{\mathbb{R}}\left(\frac{\sin\xi}{\xi/2}\right)^2\frac{d\xi}{2\pi} \;=\; \frac{2}{\pi}\int_{\mathbb{R}}\left(\frac{\sin\xi}{\xi}\right)^2 d\xi$$

Thus, the desired integral is π. ///

[03.8] (*Collecting Fourier transform pairs*) Compute the Fourier transforms of

$$\chi_{[a,b]} \qquad\qquad e^{-\pi x^2} \qquad\qquad f(x)=\begin{cases} e^{-x} & (\text{for } x>0)\\ 0 & (\text{for } x\leq 0)\end{cases}$$

Discussion: The first of these is direct:

$$\widehat{\chi_{[a,b]}}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\chi_{[a,b]}(x)\;dx \;=\; \int_a^b e^{-2\pi i\xi x}\;dx \;=\; \begin{cases} \dfrac{e^{-2\pi i\xi b}-e^{2\pi i\xi a}}{-2\pi i\xi} & (\text{for } \xi\neq 0)\\ b-a & (\text{for } \xi=0)\end{cases}$$

In particular, for $a = -b$, away from the removable singularity at $\xi = 0$, this is

$$\widehat{\chi_{[-b,b]}}(\xi) \;=\; \frac{e^{-2\pi i\xi b} - e^{-2\pi i \xi(-b)}}{-2\pi i \xi} \;=\; \frac{e^{2\pi i\xi b} - e^{-2\pi i\xi}}{2i} \cdot \frac{1}{\pi\xi} \;=\; \frac{\sin 2\pi b\xi}{\pi \xi}$$

Since $\frac{\sin 2\pi b\xi}{\pi\xi}$ is *not* in $L^1(\mathbb{R})$, but *is* in $L^2(\mathbb{R})$, we define its Fourier transform (or inverse Fourier transform) *indirectly*, via either the inversion theorem, or by extending-by-continuity via Plancherel, expressing the function as an L^2 limit of L^1 functions.

The third is similarly direct:

$$\widehat{f}(\xi) \;=\; \int_0^\infty e^{-2\pi i\xi x}\, e^{-x}\, dx \;=\; \int_0^\infty e^{-(2\pi i\xi+1)x}\, dx \;=\; \Big[\frac{e^{-(2\pi i\xi+1)x}}{-(2\pi i\xi+1)}\Big]_0^\infty \;=\; \frac{1}{2\pi i\xi+1}$$

Again, the latter function is not in L^1, but is in L^2, so its Fourier transform is most conveniently defined indirectly.

The Gaussian's Fourier transform is less trivial to evaluate, but is a very important example to have in hand, with many different applications throughout mathematics. One approach is as follows. Letting $f(x) = e^{-\pi x^2}$,

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\, e^{-\pi x^2}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi(x^2+2i\xi x)}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2-\pi\xi^2}\, dx \;=\; e^{-\pi\xi^2}\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx$$

by completing the square. The unobvious claim is that the integral does not depend on ξ, and, in fact, has value 1. Perhaps the optimal approach here is to observe that the integral is equal to a complex contour integral:

$$\int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2}\, dx \;=\; \int_{i\xi-\infty}^{i\xi+\infty} e^{-\pi z^2}\, dz$$

along the line $\mathrm{Im}(z) = i\xi$. Given the good decay of the integrand as $|\mathrm{Re}(z)| \to \infty$, by Cauchy-Goursat theory, the contour can be *moved* to integration along the real line, giving

$$\int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2}\, dx \;=\; \int_{i\xi-\infty}^{i\xi+\infty} e^{-\pi z^2}\, dz \;=\; \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx \;=\; 1$$

The fact that the latter integral has value 1 comes from the usual trick involving polar coordinates:

$$\Big(\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx\Big)^2 \;=\; \int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\, dx\, dy \;=\; \int_0^{2\pi}\int_0^\infty e^{-\pi r^2}\, r\, dr\, d\theta \;=\; 2\pi \int_0^\infty e^{-\pi r^2}\, r\, dr$$

Replacing r by $\sqrt{t}$, this is

$$\pi \int_0^\infty e^{\pi t}\, dt \;=\; \pi \cdot \frac{1}{\pi} \;=\; 1$$

Thus, with the present normalization of Fourier transform and corresponding normalization of Gaussian, the Gaussian is its own Fourier transform. ///

[03.9] Give an explicit non-zero function f such that $\int_{\mathbb{R}} x^n\, f(x)\, dx = 0$, for all $n = 0, 1, 2, \ldots$.

Discussion: We choose to find a *Schwartz* function f meeting the condition, since success in finding such f in such a relatively small class of nice functions will be a stronger result than find such f in a larger class of less-nice functions.

For Schwartz (and other) functions g, $\int_{\mathbb{R}} g(x)\, dx = \widehat{g}(0)$. Thus, the requirement on f is that

$$0 \;=\; (\widehat{x^n f})(0) \;=\; (-2\pi i)^{-n} \Big(\frac{d}{dx}\Big)^n \widehat{f}(0)$$

Thus, the requirement on $f \in \mathscr{S}$ is equivalent to the vanishing of all derivatives of $\widehat{f}$ at 0. Taking $\widehat{f}$ to be a smooth bump function with support not including 0 would suffice, for example,

$$\widehat{f}(x) \;=\; \begin{cases} e^{1/(x-1)(x-3)} & (\text{for } 1 < x < 3) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

and then f is the inverse Fourier transform of $\widehat{f}$:

$$f(x) \;=\; \int_1^3 e^{2\pi i \xi x} e^{1/(\xi-1)(\xi-3)}\, d\xi$$

Note that f cannot be compactly supported and meet the requirement, because in that case $\widehat{f}$ is an entire (holomorphic) function (in the Paley-Wiener space), which cannot vanish to infinite order at any point (without being identically 0). ///

[03.10] Show that $\chi_{[a,b]} * \chi_{[c,d]}$ is a piecewise-linear function, and express it explicitly.

Discussion: Once enunciated, this fact (and the explicit expression) should be just a matter of book-keeping. We do assume that $a \le b$ and $c \le d$. Also, by symmetry, without loss of generality we can suppose that $|b-a| \ge |d-c|$. This is used in the treatment of cases below.

$$(\chi_{[a,b]} * \chi_{[c,d]})(x) \;=\; \int_{\mathbb{R}} \chi_{[a,b]}(x-y)\cdot \chi_{[c,d]}(y)\, dy \;=\; \int_c^d \chi_{[a,b]}(x-y)\, dy$$

$$= \int_c^d \chi_{[a-x,b-x]}(-y)\, dy \;=\; \int_{-d}^{-c} \chi_{[a-x,b-x]}(y)\, dy \;=\; \text{meas}\Big([-d,-c] \cap [a-x,b-x]\Big)$$

Looking at the cases of overlap, using $b-a \ge d-c$, this is

$$\begin{cases} 0 & (\text{for } b-x \le -d, \text{ that is, } [a-x,b-x] \text{ is to the left of } [-d,-c]) \\ \\ (b-x)-(-d) & (\text{for } a-x \le -d \le b-x \le -c) \\ \\ (-c)-(-d) & (\text{for } a-x \le -d \le -c \le b-x, \text{ that is, } [-d,-c] \subset [a-x,b-x]) \\ \\ (-c)-(a-x) & (\text{for } -d \le a-x \le -c \le b-x) \\ \\ 0 & (\text{for } a-x \ge -c, \text{ that is, } [a-x,b-x] \text{ is to the right of } [-d,-c]) \end{cases}$$

$$= \begin{cases} 0 & (\text{for } x \ge b+d) \\ \\ b+d-x & (\text{for } \max(a+d,b+c) \le x \le b+d) \\ \\ d-c & (\text{for } a+d \le x \le b+c) \\ \\ -a-c+x & (\text{for } a+c \le x \le \min{(b+c,a+d)}) \\ \\ 0 & (\text{for } x \le a+c) \end{cases}$$

We used the fact that $b-a \ge d-c$ implies $a-c \le b-d$. It is useful to consider the special configuration

$[a, b] = [-A, A]$ and $[c, d] = [-B, B]$ with $A \geq B \geq 0$: the convolution is

$$\begin{cases} 0 & (\text{for } x \geq A+B) \\ A+B-x & (\text{for } A-B \leq x \leq A+B) \\ 2B & (\text{for } -A+B \leq x \leq A-B) \\ A+B+x & (\text{for } -A-B \leq x \leq -A+B) \\ 0 & (\text{for } x \leq -A-B) \end{cases}$$

In particular, the convolution is supported inside $[-A-B, A+B]$. Similarly, for f and g supported in $[-a, a]$ and $[-b, b]$, the convolution is supported in $[-a-b,\, a+b]$. ///

[03.11] For $f \in \mathscr{S}$, show that

$$\lim_{\varepsilon \to 0^+} f(x) * \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} \;=\; f(x)$$

Discussion: It suffices to show that the functions $\varphi_\varepsilon(x) = e^{-\pi x^2/\varepsilon}$ form an approximate identity, in a not-quite-strictest sense that their masses bunch up at 0, although it's *not* true that their supports shrink to $\{0\}$.

We know that $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$, so the integrals of the φ_ε are all 1. They are non-negative. Elementary estimates do show that, for fixed $\delta > 0$, $\int_{|x| \geq \delta} \varphi_\varepsilon \to 0$ as $\varepsilon \to 0^+$. This verifies that the φ_ε form an approximate identity in a slightly less-than-strictest sense, so the assertion holds. ///

For convenience, recall the general proof that this larger class of approximate identity ψ_ε (or sequence version ψ_n) has the property that $\int_{\mathbb{R}} f \cdot \psi_n \to 0$ for every Schwartz function f.

Proof: First, the integral is absolutely convergent. Second, given $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(0)| < \varepsilon$ for $|x - 0| < \delta$. For that ε and δ, take n large enough such that $|\int_{|x| \geq \delta} \psi_n(x)| < \varepsilon$. Using $\int_{\mathbb{R}} \psi_n = 1$,

$$\int_{\mathbb{R}} f \cdot \psi_n - f(0) \;=\; \int_{\mathbb{R}} f \cdot \psi_n - \int_{\mathbb{R}} f(0) \cdot \psi_n \;=\; \int_{\mathbb{R}} (f - f(0)) \cdot \psi_n \;=\; \int_{|x| < \delta} (f - f(0)) \cdot \psi_n + \int_{|x| \geq \delta} (f - f(0)) \cdot \psi_n$$

Estimate

$$\int_{|x| \geq \delta} |f - f(0)| \cdot \psi_n \;\leq\; (\sup|f| + |f(0)|) \cdot \int_{|x| \geq \delta} \psi_n \;<\; (\sup|f| + |f(0)|) \cdot \varepsilon$$

and

$$\int_{|x| < \delta} |f - f(0)| \cdot \psi_n \;\leq\; \delta \cdot \int_{|x| < \delta} \psi_n \;<\; \varepsilon \cdot (1 - \varepsilon) \;\leq\; \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, $\int f \cdot \psi_n \to f(0)$. ///

[03.12] (Corrected!) For $f \in \mathscr{S}$, show that

$$\lim_{t \to +\infty} f(x) * \frac{\sin 2\pi t x}{\pi x} \;=\; f(x)$$

Discussion: In contrast to the previous example, the functions $\varphi_n(x) = \frac{\sin 2\pi n x}{\pi x}$ (related to the Fourier-Dirichlet kernel) do *not* form an approximate identity in a straightforward sense, since they are not non-negative. And they are not in $L^1(\mathbb{R})$, so the integrals for their Fourier transforms do not converge absolutely.

But they are in $L^2(\mathbb{R})$, so *do* have Fourier transforms in the extended Fourier-Plancherel sense, and the identity $\widehat{f * \varphi_n} = \widehat{f} \cdot \widehat{\varphi}_n$ still holds. By Fourier inversion, $\widehat{\varphi}_n = \chi_{[-t,t]}$. In particular, $\widehat{f} \cdot \chi_{[-t,t]}$ converges in $L^2(\mathbb{R})$ to $\widehat{f}$ (and $\widehat{f}$ is certainly in L^2, because it is in $\mathscr{S}$).

Plancherel shows that the Fourier(-Plancherel) map and inverse are isometric isomorphisms $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, so

$$f \;=\; (\,\widehat{f}\,)^\vee \;=\; \big(L^2 - \lim_n \widehat{f} \cdot \chi_{[-t,t]}\big)^\vee \;=\; L^2 - \lim_n \Big(\big(\widehat{f} \cdot \chi_{[-t,t]}\big)^\vee\Big)$$
$$=\; L^2 - \lim_n \Big((\,\widehat{f}\,)^\vee \;*\; \chi^\vee_{[-t,t]}\Big) \;=\; f * \frac{\sin 2\pi t x}{\pi x}$$

as claimed. ///

[03.13] Evaluate the *Borwein integral*

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\, dx$$

Discussion: View this as an inner product and invoke Plancherel:

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\, dx \;=\; \Big\langle \frac{\sin x}{x},\, \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \Big\rangle \;=\; \Big\langle \Big(\frac{\sin x}{x}\Big)^{\widehat{}},\, \Big(\frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\Big)^{\widehat{}} \Big\rangle$$

Since Fourier transform converts pointwise multiplication to convolution, this is

$$\Big\langle \Big(\frac{\sin x}{x}\Big)^{\widehat{}},\, \Big(\frac{\sin x/3}{x/3}\Big)^{\widehat{}} * \Big(\frac{\sin x/5}{x/5}\Big)^{\widehat{}} \Big\rangle$$

We have computed that

$$\widehat{\chi_{[-a,a]}}(\xi) \;=\; \frac{\sin 2\pi a \xi}{\pi \xi} \;=\; 2a \cdot \frac{\sin 2\pi a \xi}{2\pi a \xi}$$

That is, by linearity of Fourier transform,

$$\Big(\frac{1}{2a}\chi_{[-a,a]}\Big)^{\widehat{}}(\xi) \;=\; \frac{\sin(2\pi a)\xi}{(2\pi a)\xi}$$

By Fourier inversion, noting that $\frac{\sin x}{x}$ is not in L^1, only in L^2, so the inverse transform is not necessarily the literal integral,

$$\Big(\frac{\sin(2\pi a)\xi}{(2\pi a)\xi}\Big)^{\widehat{}}(x) \;=\; \frac{1}{2a}\chi_{[-a,a]}(x)$$

Replacing a by $a/2\pi$ gives

$$\Big(\frac{\sin a\xi}{a\xi}\Big)^{\widehat{}}(x) \;=\; \frac{\pi}{a}\,\chi_{[-\frac{a}{2\pi},\,\frac{a}{2\pi}]}(x)$$

We will use $a = 1, \frac{1}{3}$, and $\frac{1}{5}$. The relevant convolution was also computed above, but all we need is the fact that the support of

$$3\pi\,\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * 5\pi\,\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}$$

is inside the interval $[-\frac{1}{6\pi} - \frac{1}{10\pi},\, \frac{1}{6\pi} + \frac{1}{10\pi}]$. Thus, the integral of three *sinc* functions is equal to

$$\int_{\mathbb{R}} \pi\chi_{[\frac{-1}{2\pi},\frac{1}{2\pi}]}(x) \cdot \Big(3\pi\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * 5\pi\,\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\, dx \;=\; \pi \cdot 3\pi \cdot 5\pi \int_{-1/\pi}^{1/\pi} \Big(\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * \chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\, dx$$

$$=\; \pi \cdot 3\pi \cdot 5\pi \int_{\mathbb{R}} \Big(\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * \chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\, dx$$

since $[-1/2\pi,\ 1/2\pi]$ contains the support of the convolution. Observing that (invoking Fubini-Tonelli as necessary),

$$\int_{\mathbb{R}} (f*g)(x)\ dx \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)\ dx\, dy \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\ dx\, dy \;=\; \int_{\mathbb{R}} f(x)\ dx \cdot \int_{\mathbb{R}} g(x)\ dy$$

the integral of the convolution is

$$\int_{\mathbb{R}} \chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} \cdot \int_{\mathbb{R}} \chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]} \;=\; \frac{1}{3\pi}\cdot\frac{1}{5\pi}$$

Thus, the whole is

$$\pi \cdot 3\pi \cdot 5\pi \cdot \frac{1}{3\pi}\cdot\frac{1}{5\pi} \;=\; \pi$$

Similarly, the integral of $f_1 * \ldots f_n$ is the product of the integrals $\int f_i$. With the support of f_i inside $[-a_i, a_i]$, the support of the convolution is inside $[-a_1 - \ldots - a_n,\ a_1 + \ldots + a_n]$. Thus, since $\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{13} < 1$, the same argument shows that

$$\int_{\mathbb{R}} \frac{\sin x}{x}\cdot\frac{\sin x/3}{x/3}\cdot\frac{\sin x/5}{x/2n+1}\ dx \;=\; \pi \qquad\qquad (\text{for } 2n+1 = 3,5,7,9,11,13)$$

but for $2n+1 = 15$, the support of the Fourier transform of $\frac{\sin x}{x}$ no longer contains the support of the convolution. ///
