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Examples: discussion 04

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[04.1] Give a *persuasive* proof that the function

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this to make a *smooth step function*: 0 for $x \le 0$ and 1 for $x \ge 1$, and goes monotonically from 0 to 1 in the interval $[0, 1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1, 2, 3, \ldots\}$: for each n, $f_n(x) = 1$ for $x \in [-n, n]$, $f_n(x) = 0$ for $x \notin [-(n+1), n+1]$, and f_n goes monotonically from 0 to 1 in $[-(n+1), -n]$ and monotonically from 1 to 0 in $[n, n+1]$.

Discussion: In $x > 0$, by induction, the derivatives are finite linear linear combinations of functions of the form $x^{-n}e^{-1/x}$. It suffices to show that $\lim_{x\to 0^+} x^{-n}e^{-1/x} = 0$. Equivalently, that $\lim_{x\to+\infty} x^n e^{-x} = 0$, which follows from $e^{-x} = 1/e^x$, and

$$x^{-n}e^{-1/x} \;=\; \frac{x^n}{e^x} \;=\; \frac{x^n}{\sum_{m\ge 0} \frac{x^m}{m!}} \;\le\; \frac{x^n}{\frac{x^{n+1}}{(n+1)!}} \;\longrightarrow\; 0 \qquad\qquad (\text{as } x \to +\infty)$$

(This is perhaps a little better than appeals to L'Hospital's Rule.) Thus, f is smooth at 0, with all derivatives 0 there. ///

Next, we make a *smooth bump function* by

$$b(x) \;=\; \begin{cases} 0 & (\text{for } x \le -1) \\ e^{\frac{1}{x^2-1}} & (\text{for } -1 < x < 1) \\ 0 & (\text{for } x \ge 1) \end{cases}$$

A similar argument to the previous shows that this is smooth. Renormalize it to have integral 1 by

$$\beta(x) \;=\; \frac{b(x)}{\int_{-1}^{1} b(t)\,dt}$$

Then $s(x) = \int_{-1}^{x} \beta(t)\,dt$ is a smooth (monotone) step function that goes from 0 at -1 to 1 at 1. The minor modification $s(x) = 2\int_{-1}^{x} \beta(2t-1)\,dt$ gives a smooth (monotone) step $s(x)$ function going from 0 at 0 to 1 at 1. ///

Then $s(x+n+1)$ is a smooth, monotone step function going *up* from 0 to 1 in $[-n-1, -n]$, and $s(n+1-x)$ for $n \in \mathbb{Z}$ is a smooth, monotone step function going *down* from 1 to 0 in $[n, n+1]$. Thus, the product $f_n(x) = s(x+n+1)\cdot s(n+1-x)$ is the desired smooth cut-off function. ///

[04.2] Use a family of smooth cut-offs to show that test functions on $\mathbb{R}$ are dense in Schwartz functions on $\mathbb{R}$.

Discussion: Let $s_n(x)$ be a smooth cut-off function, identically 0 for $|x| \ge n+1$, identically 1 for $|x| \le n$, real-valued, and going smoothly from 0 to 1 (or vice-versa) in the two edge-case intervals $[-n-1, -n]$ and $[n, n+1]$. Given a Schwartz function f, we claim that the pointwise products $s_n \cdot f$ are test functions converging in $\mathscr{S}$ to f. Certainly they are test functions.

Certainly $0 \le s_n(x) \le 1$ for all $x \in \mathbb{R}$ and for all n. Thus, we esimate one family of seminorms easily:

$$\sup_{x\in\mathbb{R}}(1+x^2)^m |(s_n f)(x) - f(x)| \;=\; \sup_{|x|\ge n}(1+x^2)^m |(s_n f)(x) - f(x)| \;\le\; \sup_{|x|\ge n}(1+x^2)^m |f(x)|$$

$$=\; \sup_{|x|\ge n} \frac{1}{1+x^2}(1+x^2)^{m+1}|f(x)| \;\le\; \frac{1}{n^2}\sup_{x\in\mathbb{R}}(1+x^2)^{m+1}|f(x)|$$

and the later *sup* is independent of n. Thus, the latter expression goes to 0 at $n \to +\infty$.

The s_n are designed so that for each k, the family of k^{th} derivatives $s_n^{(k)}(x)$ has a bound independent of n, for all $x \in \mathbb{R}$. Also, k^{th} derivatives with $k \ge 1$ are identically 0 outside the bands $n \le |x| \le n+1$. To warm up, estimate just the first derivative on its own:

$$\sup_{x\in\mathbb{R}}(1+x^2)^m \cdot |(s_n f)'(x) - f'(x)| \;\le\; \sup_{x\in\mathbb{R}}(1+x^2)^m \cdot |s_n' f(x) + s_n f'(x) - f'(x)|$$

$$\le \sup_{n\le|x|\le n+1}(1+x^2)^m \cdot |s_n' f(x)| + \sup_{|x|\ge n}(1+x^2)^m \cdot |f'(x)|$$

$$\le\; \sup |s_n'(x)| \cdot \sup_{|x|\ge n}(1+x^2)^m \cdot |f(x)| + \sup_{|x|\ge n}(1+x^2)^m \cdot |f'(x)|$$

As in the previous computation,

$$\sup_{|x|\ge n}(1+x^2)^m \cdot |f'(x)| \;\le\; \sup_{|x|\ge n}\frac{1}{1+x^2}(1+x^2)^{m+1}\cdot|f'(x)| \;\le\; \frac{1}{n^2}\sup_{|x|\ge n}(1+x^2)^{m+1}\cdot|f'(x)|$$

which goes to 0 as $n \to +\infty$.

The k^{th} derivatives are similarly estimated, applying Leibniz' rule k times. Thus, $\nu(s_n f - f) \to 0$ for all the seminorms ν on $\mathscr{S}$. ///

[04.3] Show that multiplication by x, and also differentiation d/dx, are continuous linear maps $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

Discussion: By the definition of the topology on $\mathscr{S}$, and by the equivalent of continuity at 0 and continuity for linear maps, it suffices to show that, given $\varepsilon > 0$ and seminorm ν on $\mathscr{S}$, there is $\delta > 0$ and seminorm μ on Schwartz functions φ such that $\mu(\varphi) < \delta$ implies $\nu(\varphi) < \varepsilon$.

For

$$\nu(\varphi) \;=\; \sup_{0\le i\le k}\sup_{x\in\mathbb{R}}(1+x^2)^n \cdot |f^{(i)}(x)|$$

we have

$$|(xf)^{(i)}(x)| \;=\; |(f+xf')^{(i-1)}(x)| \;=\; |(2f'+xf'')^{(i-2)}(x)| \;=\; \ldots \;=\; |(if^{(i-1)}+xf^{(i)})(x)|$$

$$\le\; k \cdot |f^{(i-1)}(x)| + (1+x^2)\, f^{(i)})(x)|$$

by induction. Thus,

$$\nu(xf) \;=\; \sup_{0\le i\le k}\sup_{x\in\mathbb{R}}(1+x^2)^n \cdot |f^{(i)}(x)| \;\le\; \sup_{0\le i\le k} k\sup_{x\in\mathbb{R}}(1+x^2)^n |f^{(i-1)}(x)| + \sup_{x\in\mathbb{R}}|(1+x^2)^{n+1}\, f^{(i)})(x)|$$

$$\le\; 2k \sup_{0\le i\le k}\sup_{x\in\mathbb{R}}|(1+x^2)^{n+1}\, f^{(i)})(x)|$$

So if we make

$$\mu(f) \;=\; \sup_{0\le i\le k}\sup_{x\in\mathbb{R}}|(1+x^2)^{n+1}\, f^{(i)})(x)|$$

smaller than $\varepsilon/2k$, then $|\nu(xf)| < \varepsilon$, giving the continuity of multiplication by x. ///

Even more simply, $|(f')^{(i)}(x)| = |f^{(i+1)}(x)|$ gives

$$\nu(f') \;=\; \sup_{0\le i\le k}\sup_{x\in\mathbb{R}}(1+x^2)^n\cdot|(f')^{(i)}(x)| \;=\; \sup_{1\le i\le k+1}\sup_{x\in\mathbb{R}}(1+x^2)^n\cdot|f^{(i)}(x)| \;\le\; \sup_{0\le i\le k+1}\sup_{x\in\mathbb{R}}(1+x^2)^n\cdot|f^{(i)}(x)|$$

So if we make

$$\mu(f) \;=\; \sup_{0\le i\le k+1}\sup_{x\in\mathbb{R}}|(1+x^2)^{n+1}\,f^{(i)})(x)|$$

smaller than ε, then $|\nu(f')| < \varepsilon$. ///

[04.4] Show that $\delta(\varphi) = \varphi(0)$ is a tempered distribution.

Discussion: By the definition of the topology on $\mathscr{S}$, and by the equivalence of continuity at 0 and continuity for linear maps, it suffices to show that, given $\varepsilon > 0$, there is $\eta > 0$ and seminorm ν on Schwartz functions φ such that $\nu(\varphi) < \eta$ implies $|\varphi(0)| < \varepsilon$. This succeeds for $\nu(\varphi) = \sup_{x\in\mathbb{R}}|\varphi(x)|$ and $\eta = \varepsilon$. ///

[04.5] Show that the principal value integral $\lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon}\frac{f(x)}{x}dx$ is a tempered distribution, and satisfies $x\cdot u = 1$.

Discussion: Let u be that functional. For fixed $\varepsilon > 0$, integrate by parts.The boundary terms at $\pm N$ for N large go to 0 because f is Schwartz. Boundary terms at $\pm\varepsilon$ are

$$u(f) \;=\; \lim_{\varepsilon\to 0^+}\Big(\Big[\log|x|\cdot f(x)\Big] - \int_{|x|>\varepsilon} f'(x)\cdot\log|x|\;dx\Big)$$

The boundary terms

$$\log|\varepsilon|\cdot f(\varepsilon) \;-\; \log|-\varepsilon|\cdot f(-\varepsilon) \;=\; (2\varepsilon\cdot\log\varepsilon)\cdot\frac{f(\varepsilon)-f(-\varepsilon)}{2\varepsilon}$$

are 0: differentiability of f at 0 implies that

$$\lim_{\varepsilon\to 0^+}\frac{f(\varepsilon)-f(-\varepsilon)}{2\varepsilon} \;=\; f'(0)$$

and in particular the limit *exists*, while

$$\lim_{\varepsilon\to 0^+}\varepsilon\cdot\log\varepsilon \;=\; 0$$

Thus,

$$|u(f)| \;=\; \Big|\lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon} f'(x)\cdot\log|x|\;dx\Big| \;=\; \Big|\int_{\mathbb{R}} f'(x)\cdot\log|x|\;dx\Big|$$

Further,

$$\Big|\int_{\mathbb{R}} f'(x)(1+x^2)\cdot\frac{\log|x|}{1+x^2}\;dx\Big| \;\le\; \sup_{x\in\mathbb{R}}(1+x^2)|f'(x)|(1+x^2)\cdot\int_{\mathbb{R}}\frac{|\log|x||}{1+x^2}\;dx$$

The latter integral is a finite constant, so to make $|u(f)|$ small it suffices to make the seminorm

$$\mu(f) \;=\; \sup_{0\le i\le 1}\sup_{x\in\mathbb{R}}(1+x^2)|f^{(i)}(x)|$$

small, proving the continuity. ///

For $f\in\mathscr{S}$,

$$(x\cdot u)(f) \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon}\frac{x\cdot f(x)}{x}\;dx \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} f(x)\;dx \;=\; \int_{\mathbb{R}} f(x)\;dx \;=\; \int_{\mathbb{R}} 1\cdot f(x)\;dx \;=\; 1(\varphi)$$

thinking of 1 as the integrate-against-1 distribution, since φ is continuous at 0. Thus, $x \cdot u = 1$. ///

[04.6] Reprove $\widehat{\delta} = 1$ by approximating δ by Gaussians.

Discussion: We know that $u_n(x) = \sqrt{n} \cdot e^{-\pi n x^2}$ is an *approximate identity*, meaning that

$$u_n(f) \longrightarrow f(0) = \delta(f)$$

for every $f \in \mathscr{S}$, which is exactly to say that $u_n \to \delta$ in the weak dual topology on $\mathscr{S}^*$. Fourier transform on tempered distributions is continuous, so Fourier transform and the weak-dual-topology limit can be interchanged:

$$\widehat{\delta} = \lim_n \widehat{u_n} = \lim_n e^{-\pi x^2/n}$$

the latter by earlier computations of Fourier transforms of Gaussians. By Lebesgue dominated convergence, for $f \in L^1$,

$$\lim_n \int_{\mathbb{R}} e^{-\pi x^2/n} \cdot f(x)\, dx = \int_{\mathbb{R}} \lim_n e^{-\pi x^2/n} \cdot f(x)\, dx = \int_{\mathbb{R}} 1 \cdot f(x)\, dx = 1(f) \qquad \text{(for all } f \in \mathscr{S}\text{)}$$

That is, $\widehat{\delta} = 1$. ///

[04.7] Show that $\lim_n \frac{1}{1+(x-n)^2} = 0$ in $\mathscr{S}(\mathbb{R})^*$.

Discussion: It is implicit in the question that the functionals are *integrate against* the functions $\frac{1}{1+(x-n)^2}$. By definition of the weak dual topology on $\mathscr{S}^*$, we must show that for every $f \in \mathscr{S}(\mathbb{R})$

$$\lim_n \int_{\mathbb{R}} \frac{1}{1+(x-n)^2} \cdot f(x)\, dx = 0$$

We could do a direct computation. Also, we can observe that the functions $g_n(x) = 1/(1+(x-n)^2)$ go to 0 pointwise, and for a given Schwartz function f the functions $g_n \cdot f$ go to 0 pointwise, and are dominated by $|f|$. Thus, by Lebesgue's dominated convergence theorem, the limit of the integrals is the integral of the pointwise limit, which is 0. ///

[04.8] Determine the constant c such that $x^2 \delta'' = c \cdot \delta$.

Discussion: Compute directly: for $f \in \mathscr{S}$,

$$(x^2\delta'')(f) = \delta''(x^2 \cdot f) = -\delta'(2x \cdot f + x^2 \cdot f') = \delta(2 \cdot f + 4x \cdot f' + x^2 f'')$$

$$= 2 \cdot f(0) + 4 \cdot 0 \cdot f'(0) + 0^2 \cdot f''(0) = 2 \cdot f(0) = 2\delta(f)$$

So $x^2\delta'' = 2\delta$. ///

[04.9] Show that $\sin(nx) \to 0$ in the $\mathscr{S}^*$-topology as $n \to +\infty$.

Discussion: One approach: for $f \in \mathscr{S}$, certainly $f \in L^1(\mathbb{R})$, so by Riemann-Lebesgue the Fourier transform is continuous and goes to 0 at infinity. The integral of f against $\sin(nx)$ is the imaginary part of $\widehat{f}(n)$, so goes to 0. This satisfies the definition of $\sin(nx) \to 0$ in the weak dual topology. ///

Less conceptually, but perhaps more simply, we can integrate by parts, with boundary terms going to 0 since $f \in \mathscr{S}$:

$$\int_{\mathbb{R}} f(x) \sin(nx)\, dx = -\int f'(x) \frac{-\cos(nx)}{n}\, dx$$

In absolute value, this is estimated by

$$\int |f'(x)| \, \frac{1}{n} \, dx \;=\; \frac{1}{n} \cdot \int |f'(x)| \, dx$$

If we believe that $\mathscr{S} \subset L^1$, we're done. ///

Remark: Someone might be interested to see how hard it is to verify that $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $1 \leq p < +\infty$, however reasonable the purported fact seems. It is direct, by an standard, elementary, reusable device: for $f \in \mathscr{S}$,

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx \;=;\; \int_{\mathbb{R}^n} (1+|x|^2)^{p\ell} \cdot |f(x)|^p \cdot \frac{1}{(1+|x|^2)^{p\ell}} \, dx$$

for every $\ell = 1, 2, 3, \ldots$. The latter is dominated by

$$\Big(\sup_{x \in \mathbb{R}^n} (1+|x|^2)^{p\ell} |f(x)|^p \Big) \cdot \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{p\ell}} \;=\; \Big(\sup_{x \in \mathbb{R}^n} (1+|x|^2)^{\ell} |f(x)| \Big)^p \cdot \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{p\ell}}$$

For large enough ℓ, the latter integral is convergent (to a finite value). The sup inside the p^{th} power is a fixed seminorm on $\mathscr{S}$. ///

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[04.10] Solve $u'' - \lambda \cdot u = \delta$ on $\mathbb{R}$, for $\lambda \notin \mathbb{R}$.

Discussion: Take the (extended as necessary, of course) Fourier transform of both sides:

$$1 \;=\; \widehat{\delta} \;=\; (-2\pi i \xi)^2 \widehat{u} - \lambda \cdot \widehat{u} \;=\; (4\pi^2 \xi^2 - \lambda) \cdot \widehat{u}$$

Obviously we want to *divide*, and we would obtain

$$\frac{-1}{4\pi^2 \xi^2 + \lambda} \;=\; \widehat{u}$$

The elementary pointwise division is in fact not necessarily legitimate, unless we somehow know a-priori that multiplication by the $1/(4\pi^2\xi^2 + \lambda)$ makes sense on all the entities involved.

So we should specify that we are asking for $u \in \mathscr{S}^*$, which is the most generous we can be and still use Fourier transforms in a usual (but not classical) fashion. For $\lambda \notin \mathbb{R}$, the function $\frac{1}{4\pi^2\xi^2 - \lambda}$ is smooth, and it and all its derivatives are bounded, it acts on $\mathscr{S}$ by multiplication, so acts on $\mathscr{S}^*$ by duality. Thus, we really do obtain the last equality, as tempered distributions.

Letting $\psi_\xi(x) = e^{2\pi i \xi x}$, so that we do not make question-begging errors, as an equality of tempered distributions,

$$u \;=\; \widehat{u}^\vee \;=\; -\int_{\mathbb{R}} \frac{\psi_\xi \, d\xi}{4\pi^2 \xi^2 + \lambda}$$

Since the distribution $1/(4\pi^2\xi^2 - \lambda)$ is (or, properly, comes from) a function in L^1, the literal numerical integral can compute its (inverse) Fourier transform. By residues (!),

$$u(x) \;=\; -\int_{\mathbb{R}} \frac{e^{2\pi i \xi x} \, d\xi}{4\pi^2 \xi^2 + \lambda} \;=\; -\frac{e^{-\sqrt{\lambda}|x|}}{2\sqrt{\lambda}}$$

where $\sqrt{\lambda}$ is the square root of λ with $\mathrm{Re}(\sqrt{\lambda}) > 0$. ///
