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Examples discussion 05

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[05.1] Show that $\mathscr{S}^*\text{–}\lim_{\varepsilon\to 0^+} e^{-\varepsilon|x|} = 1$.

Discussion: For the limit to be as described in the weak-dual topology on $\mathscr{S}^*$ is exactly to say that, for every $f \in \mathscr{S}$,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} e^{-|x|} \cdot f(x)\, dx \;=\; \int_{\mathbb{R}} 1 \cdot f(x)\, dx \qquad \text{(numerical limits)}$$

Sure, the functions $x \to e^{-\varepsilon|x|}$ go to 1 pointwise, so in fortunate circumstances we'd expect the limit to be as indicated. However, the function do not go to 1 *uniformly* pointwise, so something more is needed.

We can either talk about adapting a Lebesgue convergence theorem to a somewhat more general limit, but still *squeezed* between sequential limits, or just do a direct estimate since we have such explicit information. For an explicit estimate, fix small $\eta > 0$, and take N large enough so that $\int_{|x|\ge N} |f(x)|\, dx < \eta$: this is possible since $\sup_{x\in\mathbb{R}}(1+x^2)\cdot|f(x)|$ is finite. We have

$$\int_{|x|\ge N} |f(x)|\, dx \;=\; \int_{|x|\ge N} (1+x^2)\cdot|f(x)|\cdot \frac{1}{1+x^2}\, dx \;\le\; \sup_{x\in\mathbb{R}}(1+x^2)\cdot|f(x)| \cdot \int_{|x|\ge B} \frac{1}{1+x^2}\, dx \;\longrightarrow\; 0$$

since the latter sup is finite. Then

$$\Big|\int_{\mathbb{R}} e^{-\varepsilon|x|}\cdot f(x)\, dx - \int_{\mathbb{R}} 1\cdot\cdot f(x)\, dx\Big| \;\le\; \int_{\mathbb{R}} (1-e^{-\varepsilon|x|})\cdot|f(x)|\, dx$$

$$=\; \int_{|x|\le N} (1-e^{-\varepsilon|x|})\cdot|f(x)|\, dx + \int_{|x|\ge N} (1-e^{-\varepsilon|x|})\cdot|f(x)|\, dx$$

With fixed N and η, take $\varepsilon > 0$ small enough so that $1 - e^{-\varepsilon|x|} < \eta/2N$ for all $|x| \le N$. Then

$$\int_{|x|\le N} (1-e^{-\varepsilon|x|})\cdot|f(x)|\, dx + \int_{|x|\ge N} (1-e^{-\varepsilon|x|})\cdot|f(x)|\, dx \;<\; \eta+\eta$$

This holds for every $\eta > 0$. ///

[05.2] Show that $\mathscr{S}^*\text{–}\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\varepsilon^2+x^2} = \pi\cdot\delta$.

Discussion: This is similar to the previous, with some technical differences. Again, by definition of the topology involved, we must prove that for every $f \in \mathscr{S}$

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{\varepsilon}{\varepsilon^2+x^2} f(x)\, dx \;=\; \pi\cdot f(0)$$

This example differs from the previous, in that the functions go to 0 uniformly pointwise away from 0, but do not behave uniformly near 0.

It may be best to do a change of variables in the integral, replacing x by εx, so that after simplifying the integral is

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{1}{1+x^2} f(\varepsilon x)\, dx$$

Let $C = \sup_{x\in\mathbb{R}} |f(x)|$. For large N,

$$\Big|\int_{\mathbb{R}} \frac{1}{1+x^2} f(\varepsilon x)\, dx - \pi f(0)\Big| \;\le\; \int_{|x|\le N} \frac{1}{1+x^2}\cdot|f(\varepsilon x)-f(0)|\, dx + \int_{|x|\ge N} \frac{1}{1+x^2}\cdot|f(\varepsilon x)-f(0)|\, dx$$

Take $\varepsilon > 0$ small enough so that for $|x| \le N$ we have $|f(\varepsilon x) - f(0)| < \eta$: this is possible by the continuity of f at 0. Then the previous is estimated by

$$\int_{|x|\le N} \frac{1}{1+x^2} \cdot \eta \, dx + \int_{|x|\ge N} \frac{1}{1+x^2} \cdot C \, dx$$

The first summand is dominated by η, since the integral converges. The second summand goes to 0 as $N \to +\infty$, since the integral is absolutely convergent. ///

[05.3] Show that on $\mathbb{R}$ the derivative of the distribution (integrate-against-) $\log|x|$ is the principal-value distribution $\lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} \frac{f(x)\,dx}{x}$.

Discussion: Of course the basic idea is to integrate by parts, but since $1/x$ is not absolutely integral near 0, there are complications. For example, the principal-value integral is not a literal integral.

By the continuity of every $f \in \mathscr{S}$, and by the rapid decay at $\pm\infty$, using the characterization of derivative by duality,

$$(\log|x|)'(f) \;=\; -\log|x|(f') \;=\; -\int_{\mathbb{R}} \log|x| \cdot f'(x)\,dx \;=\; -\lim_{N\to+\infty} \lim_{\varepsilon\to 0^+} \int_{\varepsilon\le|x|\le N} \log|x| \cdot f'(x)\,dx$$

The (very proper!) integral can be integrated-by-parts:

$$\int_{\varepsilon\le|x|\le N} \log|x| \cdot f'(x)\,dx \;=\; [\log|x| \cdot f(x)]_{-N}^{-\varepsilon} + [\log|x| \cdot f(x)]_{\varepsilon}^{N} - \int_{\varepsilon\le|x|\le N} \frac{f(x)}{x}\,dx$$

The first two summands are

$$[\log|x| \cdot f(x)]_{-N}^{-\varepsilon} + [\log|x| \cdot f(x)]_{\varepsilon}^{N} \;=\; \log|-\varepsilon| \cdot f(-\varepsilon) - \log|-N| \cdot f(-N) + \log|N| \cdot f(N) - \log|\varepsilon| \cdot f(\varepsilon)$$

Since f is Schwartz, the evaluations at $\pm N$ go to 0. The ε terms combine to

$$\log|-\varepsilon| \cdot f(-\varepsilon) - \log|\varepsilon| \cdot f(\varepsilon) \;=\; \log\varepsilon \cdot (f(-\varepsilon) - f(\varepsilon))$$

Since f is continuously differentiable at 0, this also goes to 0. Thus, noticing that the signs cancel,

$$(\log|x|)'(f) \;=\; \lim_{N} \lim_{\varepsilon\to 0^+} \int_{\varepsilon\le|x|\le N} \frac{f(x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+} \int_{\varepsilon\le|x|} \frac{f(x)}{x}\,dx$$

since f is rapidly decreasing at $\pm\infty$. ///

[05.4] Show that the sequence $u_n = \sum_{0\le k\le n} \frac{\delta^{(k)}}{k!}$ for $n = 0, 1, 2, \ldots$ does *not* converge in $\mathcal{D}^*$.

Discussion: By the Peano-Borel theorem, there is a test function f with arbitrary Taylor-Maclaurin coefficients at 0, so we can take $f^{(n)}(0) = (-1)^n \cdot n!$ (and many easy variants). Then $u_n(f) = n+1$, which is not Cauchy in $\mathbb{C}$. If u_n were Cauchy in the weak-dual topology in $\mathcal{D}^*$, all sequences $u_n(f)$ would be Cauchy in $\mathbb{C}$, so we have a contradiction. ///

[05.5] Show that the characteristic function of an interval is in $H^{\frac{1}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{1}{2}}(\mathbb{R})$.

Discussion: By the spectral characterization of the $H^s(\mathbb{R})$-norm,

$$|\chi_{[a,b]}|^2_{H^s} \;=\; \int_{\mathbb{R}} (1+x^2)^s \cdot |\widehat{\chi}_{[a,b]}(x)|^2\,dx \;=\; -\int_{\mathbb{R}} (1+x^2)^s \cdot \Big|\frac{e^{-2\pi ibx} - e^{-2\pi iax}}{-2\pi ix}\Big|^2\,dx$$

Vanishing in the numerator of the latter fraction compensates for the vanishing in the denominator, and the integrand is continuous at 0. Thus, convergence of that integral is equivalent to convergence of the corresponding integral over $|x| \ge 1$. Since $|e^{it}| = 1$ for all real t, for $|x| \ge 1$ we can estimate

$$\int_{|x|\ge 1} (1+x^2)^s \cdot \Big|\frac{e^{-2\pi ibx} - e^{-2\pi iax}}{-2\pi ix}\Big|^2 \, dx \;\le\; \int_{|x|\ge 1} (1+x^2)^s \cdot \Big|\frac{2}{x}\Big|^2 \, dx$$

For $s < \frac{1}{2}$,

For $|x| \ge 1$, for all $s \in \mathbb{R}$, we have $(1+x^2)^s \le C \cdot x^{2s}$ with a constant C depending on s. Thus, up constants, the previous expression is dominated by

$$\int_{|x|\ge 1} |x|^{2s} \cdot \frac{1}{x^2} \, dx$$

This is convergent if $2s - 2 < -1$, which is $s < -\frac{1}{2}$.

The converse is somewhat more delicate, since the crudest estimates will not suffice to obtain an adequate lower bound. Specifically, we need something like a re-expression of the difference of two exponentials in terms of *sine*: one iconic rearrangement is

$$e^{-2\pi ibx} - e^{-2\pi iax} \;=\; e^{-2\pi i\frac{a+b}{2}x} \cdot (e^{2\pi i\frac{a-b}{2}x} - e^{-2\pi i\frac{a-b}{2}x}) \;=\; e^{-2\pi i\frac{a+b}{2}x} \cdot 2i \cdot \sin(2\pi\frac{a-b}{2}x)$$

When $2\pi\frac{a-b}{2}x$ is within $\frac{\pi}{6}$ of $n \cdot (2\pi + \frac{\pi}{2})$ for integer n, the absolute value of *sine* is at least $\frac{1}{2}$. Thus,

$$\int_{\mathbb{R}} (1+x^2)^s \cdot \Big|\frac{e^{-2\pi ibx} - e^{-2\pi iax}}{-2\pi ix}\Big|^2 \, dx \;\ge\; \sum_{n\in\mathbb{Z}} \int_{n\cdot(2\pi+\frac{\pi}{2})-\frac{\pi}{6}}^{n\cdot(2\pi+\frac{\pi}{2})+\frac{\pi}{6}} (1+x^2)^s \cdot \Big|\frac{\frac{1}{2}}{2\pi x}\Big|^2 \, dx \;\ge\; C \cdot \sum_{n=1}^{\infty} n^{2s} \cdot \frac{1}{n^2}$$

for some (finite, non-zero) constant C depending on s whose precise value is not of interest. The integral test shows that the latter expression is $+\infty$ for $s \ge -\frac{1}{2}$. ///

[05.6] Show that $f(x) = e^{-|x|}$ is in $H^{\frac{3}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{3}{2}}(\mathbb{R})$.

Discussion: Use the spectral characterization, and the fact that (up to some inessential constants) the Fourier transform of f is $\frac{1}{1+x^2}$. The H^s norm of this would be

$$\int_{\mathbb{R}} (1+x^2)^s \cdot \Big|\frac{1}{1+x^2}\Big|^2 \, dx$$

Again, it is easy to see that the integrand is essentially $|x|^{2s-4}$ for large $|x|$, giving the finiteness for $s < -\frac{3}{2} - \varepsilon$ for any/all $\varepsilon > 0$. In this example, the converse is easier that the previous. ///

[05.7] Evaluate $(\Delta - 1)e^{-|x|}$ on $\mathbb{R}$.

Discussion: This can be computed painstakingly by using the duality definition of the operation of $\Delta - 1$ on distributions. We can also execute the computation at a more desirable intuitive level (after some practice runs via the duality/integration-by-parts characterization), as follows. Away from 0, $e^{-|x|}$ is smooth, and the operation of $\Delta - 1$ can be applied in classical terms, giving 0. So the outcome will be supported at 0, and must be some finite linear combination of derivatives of δ.

The first derivative of $e^{-|x|}$ is $-(\mathrm{sgn}x) \cdot e^{-|x|}$. This introduces a jump of height -2 at 0, so (!!!) the second derivative is $-2\delta + e^{-|x|}$. Then $\Delta - 1$ applied to $e^{-|x|}$ gives -2δ. ///

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[05.8] Show that if a tempered distribution u on $\mathbb{R}^n$ satisfies $\Delta u = 0$, then u is (integrate-against-) a *polynomial.* (*This is a stronger form of Liouville's theorem from complex analysis.*)

Discussion: Suppose $\Delta u = 0$. Taking Fourier transform (and dividing through by the irrelevant non-zero constant), we obtain $r^2 \cdot \widehat{u} = 0$. We have a result that says that the tempered distribution $\widehat{u}$ must have support inside the zero-set of r^2, which is $\{0\}$. We also have the classification of such distributions supported in $\{0\}$, namely, finite linear combinations of δ and and its derivatives. Fourier transforms of these are polynomials, so u must have been a polynomial (annihilated by Δ). ///

[05.9] Show that d/dx is a continuous operator on $C^\infty(\mathbb{T})$, where $\mathbb{T}$ is the circle $\mathbb{R}/\mathbb{Z}$.

Discussion:

[05.10] Let $\psi_n(x) = e^{inx}$. Show that $\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$ converges in the Sobolev space $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$.

Discussion: The spectral characterization requires exactly that we show that

$$\sum_n (1+n^2)^s \cdot |1|^2 \;<\; +\infty$$

for $s < -\frac{1}{2}$. This follows from the integral test. ///

[05.11] Differentiate $\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$ twice.

Discussion: The infinite sum converges in H^s for $s < -\frac{1}{2} - \varepsilon$, by the previous example. Differentiation maps $H^s \to H^{s-1}$ continuously, for all s. Thus, since infinite sums are limits of the corresponding finite sums, in the ambient topology, we can always differentiate termwise. Thus,

$$\frac{d}{dx}\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n \;=\; \sum_{n\in\mathbb{Z}} 1 \cdot \frac{d}{dx}\psi_n \;=\; \sum_{n\in\mathbb{Z}} 1 \cdot 2\pi i n \cdot \psi_n \qquad \text{(convergent in } H^{-\frac{1}{2}-\varepsilon-1}\text{)}$$

Repeating,

$$\frac{d}{dx}\sum_{n\in\mathbb{Z}} 1 \cdot 2\pi i n \psi_n \;=\; \sum_{n\in\mathbb{Z}} 1 \cdot 2\pi i n \cdot \frac{d}{dx}\psi_n \;=\; \sum_{n\in\mathbb{Z}} 1 \cdot (2\pi i n)^2 \cdot \psi_n \qquad \text{(convergent in } H^{-\frac{1}{2}-\varepsilon-2}\text{)}$$

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[05.12] Find a continuous function f on $\mathbb{T}$ such that $f'' - f = \sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$.

Discussion: The distribution $u = \sum_{n\in\mathbb{Z}} 1 \cdot \psi_n$ is in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ for every $\varepsilon > 0$, by the spectral characterization. The differential equation $(\Delta - 1)f = u$ can be solved *by division* for f having a Fourier expansion, but equating Fourier coefficients (using uniqueness of Fourier expansions in all Sobolev spaces H^s): $((2\pi i n)^2 - 1)\widehat{f}(n) = \widehat{u}(n) = 1$. Thus, $\widehat{f}(n) = -1/(4\pi^2 n^2 + 1)$. Either by direct computation, or by anticipation, $f \in H^{-\frac{1}{2}-\varepsilon+2} \subset H^{\frac{1}{2}+\varepsilon}$. By Sobolev imbedding, this is inside $C^o(\mathbb{T})$. Thus, the f specified by this Fourier expansion is continuous. ///

[05.13] Classify distributions u on $\mathbb{R}^2$ such that $r^2 \cdot u = 0$, where r is radius.

Discussion: We have a theorem that says that u must be supported on the 0-set of r^2, which is just $\{0\}$. We also have a theorem that says that the only distributions supported at 0 are finite linear combinations of derivatives of δ.

Further, taking Fourier transform, we have $\Delta\widehat{u} = 0$, where (as we already know) $\widehat{u}$ is a polynomial. This condition holds if and only if $\widehat{u}$ is a *harmonic* polynomial, that is, a polynomial annihilated by Δ.

In $\mathbb{R}$, the only such polynomials are linear or constants. However, in $\mathbb{R}^2$, all the polynomials $(x \pm iy)^n$ are harmonic. (One can also prove that *all* harmonic polynomials on $\mathbb{R}^2$ are finite linear combinations of these.) In particular, there are infinitely many linearly-independent such u. ///

[05.14] Show that $u \in H^{-1}(\mathbb{R})$ is $f'' - f$ for some continuous function f.

Discussion: By Fourier transform, assuming $f \in \mathscr{S}^*$, the equation $f''-f = u$ becomes $((-2\pi i x)^2 - 1)\widehat{f} = \widehat{u}$. Since $(-2\pi ix)^2 - 1$ does not vanish, and is smooth, of polynomial growth/decay, we can divide, giving

$$\widehat{f} \;=\; \frac{\widehat{u}}{-(4\pi^2 x^2 + 1)}$$

The usual estimate shows that $f \in H^{-1} + 2 = H^1$. By Sobolev imbedding, $H^1 \subset H^{\frac{1}{2}+\varepsilon} \subset C^o$, so f is also continuous. ///

[05.15] Show that the (distributional) derivative of a *finite* positive, regular Borel measure μ on $\mathbb{T}$ is in $H^{-\frac{3}{2}-\varepsilon}(\mathbb{T})$ for every $\varepsilon > 0$. (Hint: Riesz-Markov-Kakutani theorem.)

Discussion: By R-M-K, the collection of (regular, finite, Borel) measures on $\mathbb{T}$ is the dual $C^o(\mathbb{T})^*$ of continuous functions. By the simplest case of Sobolev imbedding, for every $\varepsilon > 0$, $H^{\frac{1}{2}+\varepsilon}(\mathbb{T}) \subset C^o(\mathbb{T})$. Thus, dualizing (and using the density of that Sobolev space in continuous functions to have injectivity), we have an injection

$$C^o(\mathbb{T})^* \;\subset\; (H^{\frac{1}{2}+\varepsilon}(\mathbb{T})^* \;=\; H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$$

The (extended) derivative maps $H^s \to H^{s-1}$ continuously, so the distributional derivative (not Radon-Nikodym!), $\frac{d}{dx}\mu$ is inside

$$\frac{d}{dx} H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \;=\; H^{-\frac{1}{2}-\varepsilon-1}(\mathbb{T}) \;=\; H^{-\frac{3}{2}-\varepsilon}(\mathbb{T})$$

as claimed. ///

[05.16] Prove the *Sokhotski-Plemelj formula* (often arising in physics settings): for Schwartz f,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{f(x)\,dx}{x \pm i\varepsilon} \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} \frac{f(x)\,dx}{x} \;\mp\; \pi i \cdot f(0)$$

(*Hint:* the earlier example of the limiting behavior of $\varepsilon/(\varepsilon^2 + x^2)$ is helpful.)

Discussion: From the earlier computation,

$$\int_{\mathbb{R}} \frac{f(x)\,dx}{x \pm i\varepsilon} \;=\; \int_{\mathbb{R}} \frac{f(x)\,(x \mp i\varepsilon)\,dx}{x^2+\varepsilon^2} \;=\; \int_{\mathbb{R}} \frac{f(x)\,x\,dx}{x^2+\varepsilon^2} \mp i \int_{\mathbb{R}} \frac{f(x)\,\varepsilon\,dx}{x^2+\varepsilon^2} \;=\; \int_{\mathbb{R}} \frac{f(x)\,x\,dx}{x^2+\varepsilon^2} \mp \pi i f(0)$$

Of course,

$$\int_{\mathbb{R}} \frac{f(x)\,x\,dx}{x^2+\varepsilon^2} \;=\; \int_{|x|\le\varepsilon} \frac{f(x)\,x\,dx}{x^2+\varepsilon^2} + \int_{|x|\ge\varepsilon} \frac{f(x)\,x\,dx}{x^2+\varepsilon^2}$$

We claim that the first of the latter two integrals goes to 0 as $\varepsilon \to 0^+$. Replacing x by $x\varepsilon$ and simplifying, it is

$$\int_{|x|\le 1} \frac{f(\varepsilon x)\,x\,dx}{x^2+1}$$

Given $\eta > 0$, we can make $\varepsilon > 0$ small enough so that $|f(x) - f(0)| < \eta$ for $|x| \le \varepsilon$, by the continuity of f at 0. With such ε,

$$\int_{|x|\le 1} \frac{f(\varepsilon x)\,x\,dx}{x^2+1} \;=\; f(0) \int_{|x|\le 1} \frac{x\,dx}{x^2+1} + \int_{|x|\le 1} \frac{(f(\varepsilon x) - f(0))\,x\,dx}{x^2+1}$$

The second of the latter two integrals is dominated by

$$\eta \cdot \int_{|x|\le 1} \frac{|x|\;dx}{1+x^2} \;\longrightarrow\; 0$$

The first of the latter two is an integral of an *odd* function against an *even* function, so is 0.

It remains to show that

$$\lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} \frac{f(x)}{x}\Big(1 - \frac{x^2}{x^2+\varepsilon^2}\Big)\;dx \;=\; 0$$

By replacing x by εx, the integral is

$$\int_{|x|\ge\varepsilon} \frac{f(x)}{x}\frac{\varepsilon^2}{x^2+\varepsilon^2}\;dx \;=\; \int_{|x|\ge 1} \frac{f(\varepsilon x)}{x}\frac{1}{x^2+1}\;dx$$

Given $\eta>0$, take N large and take $\varepsilon>0$ small enough so that $|f(\varepsilon x)-f(0)|<\eta$ for $1\le|x|\le N$. Rewrite

$$\int_{|x|\ge 1} \frac{f(\varepsilon x)}{x}\frac{1}{x^2+1}\;dx \;=\; \int_{1\le|x|\le N} \frac{f(\varepsilon x)}{x}\frac{1}{x^2+1}\;dx + \int_{|x|\ge N} \frac{f(\varepsilon x)}{x}\frac{1}{x^2+1}\;dx$$

The first of the latter two integrals is

$$\int_{1\le|x|\le N} \frac{f(\varepsilon x)}{x}\frac{1}{x^2+1}\;dx \;=\; \int_{1\le|x|\le N} \frac{f(\varepsilon x)-f(0)}{x}\frac{1}{x^2+1}\;dx + f(0)\int_{1\le|x|\le N} \frac{1}{x}\frac{1}{x^2+1}\;dx$$

The first of the latter integrals is estimated by $2N\cdot\eta$, which we can make arbitrarily small. The second integral is an integral of an odd function against an even function, so is 0. Also,

$$\int_{|x|\ge N} \frac{f(\varepsilon x)}{x}\frac{1}{x^2+1}\;dx \;\le\; \sup|f(x)|\cdot\int_{|x|\ge N}\frac{1}{|x|^3}\;dx \;\le\; 2\cdot\sup|f(x)|\cdot\frac{1}{N^2}$$

which we can make arbitrarily small. ///
