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# *Examples discussion 06*

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-disc-06.pdf]

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[06.1] On $\mathbb{T}$, show that $u'' = \delta_{\mathbb{Z}}$ has no solution $u \in \mathcal{D}^*$.

**Discussion:** Let $\psi_n(x) = e^{2\pi inx}$. We can use Fourier expansions for every element of $\mathcal{D}^* = \mathcal{D}(\mathbb{T})^*$, because by Sobolev imbedding $\mathcal{D}^* = H^{-\infty}$. Also, because differentiation is a continuous map on $H^{-\infty}$, Fourier series can always be differentiated termwise. Thus, the equation

$$\Big(\frac{d}{dx}\Big)^2\Big(\sum_n \widehat{u}(n)\psi_n\Big) \;=\; u'' \;=\; \delta_{\mathbb{Z}} \;=\; \sum_n 1\cdot\psi_n$$

is equivalent to coefficient-wise equality, which is $(2\pi in)^2\widehat{u}(n) = 1$. This is impossible for $n = 0$. ///

[06.2] Define *translation* $T_{x_o}u$ of a distribution $u$ by an amount $x_o \in \mathbb{R}$ by

$$(T_{x_o}u)(\varphi) \;=\; u(T_{-x_o}\varphi) \qquad\qquad (\text{for } \varphi \in \mathcal{D})$$

The sign is for compatibility with distributions arising as integrate-against test functions. For tempered $u$, express $\widehat{T_{x_o}u}$ in terms of $\widehat{u}$. ///

**Discussion:** Given the compatibility with ordinary Fourier transform on nice functions, taking into account the integration-against aspect, it suffices to determine the relation on those nice functions: by changing variables, replacing $x$ by $x - x_o$,

$$f(x+x_o)\widehat{\;}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\, f(x+x_o)\,dx \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi(x-x_o)}\, f(x)\,dx \;=\; e^{2\pi i\xi x_o}\cdot\widehat{f}(\xi)$$

Because we want distributions to extend integrating-against-functions, and since

$$\int_{\mathbb{R}} T_{x_o}f(x)\,F(x)\,dx \;=\; \int_{\mathbb{R}} f(x+x_o)\,F(x)\,dx \;=\; \int_{\mathbb{R}} f(x)\,F(x-x_o)\,dx$$

so the correct definition of $T_{x_o}u$ is

$$(T_{x_o}u)(\varphi) \;=\; u(T_{-x_o}\varphi)$$

Thus, in these conventions, there is a sign flip: for tempered distributions $u$,

$$\widehat{T_x u} \;=\; e^{-2\pi i\xi x_o}\cdot\widehat{u}$$

This confusion about signs seems to be inescapable. ///

[06.3] Compute $\widehat{\cos x}$.

**Discussion:** Using the previous example, letting $T_{x_o}$ be translation by $x_o$,

$$2\cos x \;=\; e^{2\pi ix}\cdot 1 + e^{-2\pi ix}\cdot 1 \;=\; e^{2\pi ix}\cdot\widehat{\delta} + e^{-2\pi ix}\cdot\widehat{\delta} \;=\; \widehat{T_x\delta} + \widehat{T_{-x}\delta}$$

By Fourier inversion (on tempered distributions, and using the fact that cosine is *even*),

$$\widehat{\cos x} \;=\; \tfrac{1}{2}\Big(T_x\delta + T_{-x}\delta\Big) \;=\; \text{"}\tfrac{1}{2}(\delta(\xi-x)+\delta(\xi+x))\text{"}$$

where the latter expresses the intent/idea of the thing, though is slightly imprecise. Note also the sign flips, which happen to have no impact on the outcome, since cosine is an even function. ///

[06.4] On $\mathbb{R}^n$, show that $|x|^2 \cdot \Delta\delta = 2n \cdot \delta$.

**Discussion:** First, for $\varphi \in \mathcal{D}$,

$$\Big(|x|^2 \cdot \Delta\delta\Big)(\varphi) \;=\; \delta(\Delta(|x|^2 \cdot \varphi)\Big)$$

By direct computation,

$$\Delta(|x|^2 \cdot \varphi) \;=\; \sum_j \Big(2 \cdot \varphi + 4x_j \frac{\partial \varphi}{\partial x_j} + r^2 \frac{\partial^2 \varphi}{\partial x_j^2}\Big)$$

Upon application of $\delta$, that is, evaluation at 0, all the terms vanish except the $2 \cdot \varphi$, which is summed from 1 to $n$, giving $2n \cdot \varphi(0) = 2n \cdot \delta(\varphi)$. ///

[06.5] Compute the Fourier transform of the *sign* function

$$\mathrm{sgn}(x) \;=\; \begin{cases} -1 & (x < 0) \\ \\ +1 & (x > 0) \end{cases}$$

**Discussion:** The sign function is *odd*, and of positive-homogeneous degree 0. Thus, by computations about the interaction of Fourier transform and Euler operator, its Fourier transform is also odd, and of degree $-(1-0)$, where the 1 is the dimension, and the 0 is the degree of the sign function.

We have seen that the principal value integral against $1/x$ is odd and of degree $-1$. The uniqueness theorem for homogeneous distributions of a given parity implies that the Fourier transform of the sign function must be a constant multiple of the principal value integral against $1/x$.

To determine the constant, apply both to an odd Schwartz function whose Fourier transform we understand, such as the iconic $xe^{-\pi x^2}$, whose Fourier transform is $-i$ times it. (Maybe later: determination of the constant is secondary.)

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[06.6] Compute the two-dimensional Fourier transform of $(x \pm iy)^n \cdot e^{-\pi(x^2+y^2)}$. (*Hint:* It is useful to rewrite things in terms of a complex variable $z = x + iy$ and its complex conjugate $\overline{z} = x - iy$.)

**Discussion:** Using the complex coordinates, Fourier transform is

$$\widehat{f}(w) \;=\; \int_{\mathbb{C}} e^{-2\pi i \mathrm{Re}\,(z\overline{w})} f(z)\,dx\,dy \;=\; \int_{\mathbb{C}} e^{-\pi i(z\overline{w} + \overline{z}w)} f(z)\,dx\,dy$$

Thus, with the plus sign in the $\pm$, since the Fourier transform of (a suitably normalized) Gaussian is itself,

$$\int_{\mathbb{C}} e^{-\pi i(z\overline{w} + \overline{z}w)} z^n e^{-\pi z\overline{z}}\,dx\,dy \;=\; \frac{1}{(-\pi i)^n} \int_{\mathbb{C}} \Big(\frac{\partial}{\partial \overline{w}}\Big)^n e^{-\pi i(z\overline{w} + \overline{z}w)} e^{-\pi z\overline{z}}\,dx\,dy$$

$$= \frac{1}{(-\pi i)^n} \Big(\frac{\partial}{\partial \overline{w}}\Big)^n \int_{\mathbb{C}} e^{-\pi i(z\overline{w} + \overline{z}w)} e^{-\pi z\overline{z}}\,dx\,dy \;=\; \frac{1}{(-\pi i)^n} \Big(\frac{\partial}{\partial \overline{w}}\Big)^n e^{-\pi w\overline{w}}$$

$$= \frac{1}{(-\pi i)^n} (-\pi w)^n \cdot e^{-\pi w\overline{w}} \;=\; i^{-n} \cdot w^n e^{-\pi w\overline{w}}$$

as claimed. Yes, one might take a moment to check that the usual symbol manipulation does extend correctly to this complex-variables variation. ///

**[06.7]** The Cauchy-Riemann operator on $\mathbb{C} \approx \mathbb{R}^2$ is

$$\overline{\partial} \;=\; \frac{\partial}{\partial \overline{z}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

Let $u_{n,s}(z) = \big(\frac{z}{|z|}\big)^n \cdot |z|^s$ for $n \in \mathbb{Z}$ and $s \in \mathbb{C}$. Determine the requirements on $n, s$ such that $u_{n,s}$ is locally integrable (and, thus, because it is of moderate growth, gives a tempered distributions). Compute $\overline{\partial} u_{n,s}$, and explain how to interpret the outcome in case the outcome is no longer locally integrable.

**Discussion:** The local integrability for $\mathrm{Re}(s) > -2$ is immediate, upon changing to polar coordinates.

Just in symbols (which, with well-chosen symbols, surely suggest the correct outcomes),

$$\overline{\partial} u_{n,s} \;=\; \overline{\partial} z^{\frac{n+s}{2}} \overline{z}^{\frac{-n+s}{2}} \;=\; \frac{-n+s}{2} \cdot z^{\frac{n+s}{2}} \overline{z}^{\frac{-n+s}{2}-1} \;=\; \frac{-n+s}{2} \cdot \Big(\frac{z}{|z|}\Big)^{n+1} \cdot |z|^{s-1}$$

As with $|x|^s$ on $\mathbb{R}^n$, we can *regularize* these distributions outside the range of local integrability by examining their behavior under $\Delta = 4\partial\overline{\partial}$. Namely,

$$\Delta u_{n,s} \;=\; (s^2 - n^2) \cdot u_{n,s-2}$$

Replacing $s$ by $s+2$ and rearranging,

$$u_{n,s} \;=\; \frac{\Delta u_{n,s+2}}{s^2 - n^2}$$

Thus, while we have local integrability in $\mathrm{Re}(s) > -2$, the right-hand side of the latter expression gives a (tempered) distribution for $\mathrm{Re}(s+2) > -2$, that is, for $\mathrm{Re}(s) > -4$. Iterating this process unambiguously defines a distribution for all $s \in \mathbb{C}$ away from $-2, -4, -6, \ldots$. ///

**[06.8]** On $\mathbb{R}^n$, for fixed $\varphi \in \mathcal{D}$, show that the function $f_\varphi(s) = \int_{\mathbb{R}^n} \varphi(x)\,|x|^s\,dx$ blows up as $s \to -n^+$, in particular, there is a constant $C_n$ such that

$$f_\varphi(s) \;=\; \frac{C_n \cdot \varphi(0)}{s+n} + (\text{continuous at } -n)$$

(Thus, if we understand that $s \to$ integration-against $|x|^s$ is a *meromorphic* distribution-valued function, its *residue* at $s = -n$ is a constant multiple of $\delta$.)

**Discussion:** Let $\psi(x) = e^{-\pi x^2}$. Then

$$f_\psi(s) \;=\; \int_{\mathbb{R}^n} \psi(x)\,|x|^s\,dx \;=\; |S^{n-1}| \cdot \int_0^\infty e^{-\pi r^2}\, r^s\, r^{n-1}\,dr \;=\; |S^{n-1}| \cdot \tfrac{1}{2} \int_0^\infty e^{-\pi r}\, r^{\frac{s+n}{2}-1}\,dr$$

$$=\; |S^{n-1}| \cdot \tfrac{1}{2}\pi^{-\frac{s+n}{2}} \int_0^\infty e^{-r}\, r^{\frac{s+n}{2}-1}\,dr \;=\; |S^{n-1}| \cdot \tfrac{1}{2}\pi^{-\frac{s+n}{2}} \cdot \Gamma(\frac{s+n}{2})$$

The Gamma function $\Gamma(z)$ has a simple pole at $z = 0$, so the latter expression blows up (in that sense) at $s + n = 0$, which is $s = -n$. More precisely,

$$\Gamma(s) \;=\; \frac{1}{s} + (\text{holomorphic at } s = 0)$$

so

$$\Gamma(\frac{s+n}{2}) \;=\; \frac{2}{s+n} + (\text{holomorphic at } s = -n)$$

and

$$f_\psi(s) \;=\; |S^{n-1}| \cdot \tfrac{1}{2} \cdot \frac{2}{s+n} \;+\; \text{(holomorphic at } s=-n) \;=\; \frac{|S^{n-1}|}{s+n} \;+\; \text{(holomorphic at } s=-n)$$

Certainly $\varphi \to f_\varphi(s)$ is *linear* in the argument $\varphi$. Since $F = \varphi - \varphi(0)\cdot\psi$ is 0 at 0, the integral for $f_F(s)$ converges absolutely in $\mathrm{Re}(s) > -n-1$. Thus,

$$f_\varphi(s) \;=\; f_F(s) + \varphi(0)\cdot f_\psi(s) \;=\; \text{(holo at } s=-n) + \varphi(0)\cdot\Big(\frac{|S^{n-1}|}{s+n} \;+\; \text{(holo at } s=-n)\Big)$$

$$=\; \frac{\varphi(0)\cdot|S^{n-1}|}{s+n} \;+\; \text{(holo at } s=-n)$$

This holds for every $\varphi$, so the residue at $s=-n$ is $|S^{n-1}|$ times $\delta$. ///

[06.9] The Riemann-equation characterizing holomorphic functions $f$ is $\overline{\partial} f = 0$. Show that

$$\overline{\partial}\,\frac{1}{z} \;=\; \text{(constant multiple of)}\;\delta$$

**Discussion:** Yes, this fact mirrors the Cauchy formulas. Taking Fourier transform,

$$-i\pi w \cdot \widehat{\frac{1}{z}} \;=\; 1$$

The function/distribution $1/z$ is positive homogeneous of degree $-1$, and rotation-equivariant by $\mu \to \mu^{-1}$. Thus, its Fourier transform is homogeneous of degree $-(2-1) = -1$, and it has the same rotation equivariance. Applying $\overline{\partial}$ gives it rotation *invariance*, and homogeneity degree $(-1)-1 = -2$. By the uniqueness theorem, up to constants, this Fourier transform must be $\delta$.

To determine the constant, apply both to a convenient test function. ///

[06.10] On $\mathbb{R}^2 \approx \mathbb{C}$, show that $Tf(z) = f(z)/z\}$ is a continuous map of the subspace $\mathscr{S}_1 = \{f \in \mathscr{S}(\mathbb{R}^2) : f(\mu z) = \mu\cdot f(z)\,\forall|\mu| = 1\}$ to $C^o(\mathbb{R}^2)$. (*Hint:* Use Taylor-Maclaurin series.)

**Discussion:** *[... iou ...]*

[06.11] On $\mathbb{R}^2 \approx \mathbb{C}$, show that the principal-value integral

$$u(f) \;=\; \lim_{\varepsilon\to 0^+} \int_{|z|\ge\varepsilon} f(z)\,\frac{z}{|z|^3}\,dx\,dy \qquad\qquad \text{(for } f \in \mathscr{S})$$

gives a tempered distribution.

**Discussion:** As in other examples, and the one-dimensional principal-value integral against $1/x$, this integral is interesting because it is at the edge of the region of local integrability of functions $z/|z|^s$, with $s \in \mathbb{C}$. Understanding of it can be construed as an instance of *regularization.*

Integrating by parts twice, using $\Delta = 4\partial\overline{\partial}$, doing the requisite subordinate estimates, this functional is

$$\lim_{\varepsilon\to 0^+} \int_{|z|\ge\varepsilon} f(z)\,\frac{z}{|z|^3}\,dx\,dy \;=\; -4\lim_{\varepsilon\to 0^+} \int_{|z|\ge\varepsilon} \Delta f(z)\,\frac{z}{|z|}\,dx\,dy \;=\; -4\int_{\mathbb{C}} \Delta f(z)\,\frac{z}{|z|}\,dx\,dy$$

since $z/|z|$ is locally integrable. Thus, it gives a distribution. Also, it is (pointwise) *bounded*, so certainly of suitably moderate growth to give a *tempered* distribution. ///

[06.12] Compute the Fourier transform of the distribution in the previous example.

**Discussion:** By either its direct definition or the equivalent *regularized* characterization from the previous example, this distribution is of homogeneous degree $-2$, and has rotation-equivariance (using the model $\mathbb{R}^2 = \mathbb{C}$) $\mu \to \mu^1$. From earlier discussion of behavior of rotation-equivariance and homogeneity under Fourier transform, up to a constant, its Fourier transform has the same rotational equivariance, and is of homogeneity degree $-(-2-(-2)) = 0$. By the uniqueness theorem, this Fourier transform is a constant multiple of the (unique-up-to-constants) degree-zero distribution $z/|z|$ with the same rotation-equivariance.

To determine the constant, evaluate things at a Schwartz function with the same rotation-equivariance, and whose Fourier transform is understood, such as $z^n e^{-\pi z\overline{z}}$, for $n \geq 0$. ... *[... iou ...]* ///

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