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Examples 01

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-01.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 27 Sept 2019.

[01.1] Let $c_1 > c_2 > c_3 > \ldots > 0$ with $c_n \to 0$. Show that, for all $0 < x < 2\pi$, $\sum_n c_n e^{inx}$ is *convergent*.

[01.2] Let f_n be the piece-wise linear tent functions with heights n, widths $2/n$, centered at $1/n$: formulaically,

$$f_n(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0 \text{ or } x \ge \frac{2}{n}) \\ n^2 \cdot x & (\text{for } 0 \le x \le \frac{1}{n}) \\ n - n^2(x - \frac{1}{n}) & (\text{for } \frac{1}{n} \le x \le \frac{2}{n}) \end{cases}$$

Show that for $g \in C^o(\mathbb{R})$,

$$\lim_n \int_{\mathbb{R}} f_n(x)\, g(x)\, dx \;=\; g(0)$$

[01.3] There is not much hope in making sense of the outcome of an uncountable number of non-zero operations: let Ω be an *uncountable* collection of positive real numbers. Letting F range over all finite subsets of Ω, show that $\sup_F \sum_{\alpha\in F} \alpha = +\infty$.

[01.4] Show that the closed unit ball in ℓ^2 is *not compact*.

[01.5] Show that the closed unit ball in $C^o[a,b]$ is not compact.

[01.6] Show that $C^o[a,b]$ is *not complete* with the $L^2[a,b]$ metric.

[01.7] Show that $C^1[a,b]$ is *not complete* with the $C^o[a,b]$ metric.

[01.8] Show that $C^1[a,b]$ is complete, with the $C^1[a,b]$ metric

$$d(f,g) \;=\; \sup_{a\le x\le b} |f(x) - g(x)| + \sup_{a\le x\le b} |f'(x) - g'(x)|$$

[01.9] Show that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \frac{1}{n}\}$$

is compact. (Hint: use the *total boundedness* criterion.)

[01.10] The space of continuous functions on $\mathbb{R}$ *going to* 0 *at infinity* is

$$C^o_o(\mathbb{R}) \;=\; \{f \in C^o(\mathbb{R}) : \text{for every } \varepsilon > 0 \text{ there is } T \text{ such that } |f(x)| < \varepsilon \text{ for all } |x| \ge T\}$$

Show that the closure of $C^o_c(R)$ in the space $C^o_{\mathrm{bdd}}(\mathbb{R})$ of bounded continuous functions with sup norm, is $C^o_o(\mathbb{R})$.
