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Examples 03

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[This document is http://www.math.umn.edu/˜garrett/m/real/notes_2019-20/real-ex-03.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 08 Nov 2019.

[03.1] For a vector subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the topological closure of W.

[03.2] Find two *dense* vector subspaces X, Y of ℓ^2 such that $X \cap Y = \{0\}$. (And, if you need further entertainment, can you find countably-many dense vector subspaces X_n such that $X_m \cap X_n = \{0\}$ for $m \neq n$?)

[03.3] For measurable $E \subset [0,1]$, show that $\lim_n \int_E e^{-2\pi i n x}\,dx = 0$ as $n \to \infty$ ranging over integers.

[03.4] Let $f_n(x) = \sin \pi n x$ on $[0,1]$, extended by $\mathbb{Z}$-periodicity, for $n = 1, 2, 3, \ldots$. Given $g \in L^1[0,1]$, show that $\int_0^1 f_n \cdot g \to 0$.

[03.5] Compute the Fourier coefficients of the sawtooth function $s(x) = x - \frac{1}{2}$ on $[0,1]$, extended by $\mathbb{Z}$-periodicity. Use this to show that $\sum_{n \geq 1} 1/n^2 = \pi^2/6$.

[03.6] Let E be a Lebesgue measurable set in $\mathbb{R}$ with finite Lebesgue measure. Show that

$$\lim_{t \to +\infty} \int_E \sin tx\,dx \;=\; 0 \qquad \text{(over real } t\text{)}$$

[03.7] Compute $\displaystyle\int_{\mathbb{R}} \Big(\frac{\sin x}{x}\Big)^2\,dx$. (*Hint:* do not attempt to do this directly, nor by complex analysis.)

[03.8] (*Collecting Fourier transform pairs*) Compute the Fourier transforms of

$$\chi_{[a,b]} \qquad\qquad e^{-\pi x^2} \qquad\qquad f(x) = \begin{cases} e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x \leq 0) \end{cases}$$

[03.9] Give an explicit non-zero function f such that $\int_{\mathbb{R}} x^n\, f(x)\,dx = 0$, for all $n = 0, 1, 2, \ldots$.

[03.10] Show that $\chi_{[a,b]} * \chi_{[c,d]}$ is a piecewise-linear function, and express it explicitly.

[03.11] For $f \in \mathscr{S}$, show that

$$\lim_{\varepsilon \to 0^+} f(x) * \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} \;=\; f(x)$$

[03.12] (Corrected!) For $f \in \mathscr{S}$, show that

$$\lim_{t \to +\infty} f(x) * \frac{\sin 2\pi t x}{\pi x} \;=\; f(x)$$

[03.13] Evaluate the *Borwein integral*

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \; dx$$
