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Examples 05

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-05.pdf]

For feedback on these examples, please get your write-ups to me by Friday, Feb 07.

[05.1] Show that $\mathscr{S}^*$–$\lim_{\varepsilon\to 0^+} e^{-\varepsilon|x|} = 1$.

[05.2] Show that $\mathscr{S}^*$–$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\varepsilon^2+x^2} = \pi\cdot\delta$.

[05.3] Show that on $\mathbb{R}$ the derivative of the distribution (integrate-against-) $\log|x|$ is the principal-value distribution $\lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} \frac{f(x)\,dx}{x}$.

[05.4] Show that the sequence $u_n = \sum_{0\le k\le n} \frac{\delta^{(k)}}{k!}$ for $n = 0,1,2,\ldots$ does *not* converge in $\mathcal{D}^*$.

[05.5] Show that the characteristic function of an interval is in $H^{\frac{1}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{1}{2}}(\mathbb{R})$.

[05.6] Show that $f(x) = e^{-|x|}$ is in $H^{\frac{3}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{3}{2}}(\mathbb{R})$.

[05.7] Evaluate $(\Delta - 1)e^{-|x|}$ on $\mathbb{R}$.

[05.8] Show that if a tempered distribution u on $\mathbb{R}^n$ satisfies $\Delta u = 0$, then u is (integrate-against-) a *polynomial.* (*This is a stronger form of Liouville's theorem from complex analysis.*)

[05.9] Show that d/dx is a continuous operator on $C^\infty(\mathbb{T})$, where $\mathbb{T}$ is the circle $\mathbb{R}/\mathbb{Z}$.

[05.10] Let $\psi_n(x) = e^{inx}$. Show that $\sum_{n\in\mathbb{Z}} 1\cdot\psi_n$ converges in the Sobolev space $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$.

[05.11] Differentiate $\sum_{n\in\mathbb{Z}} 1\cdot\psi_n$ twice.

[05.12] Find a continuous function f on $\mathbb{T}$ such that $f'' - f = \sum_{n\in\mathbb{Z}} 1\cdot\psi_n$.

[05.13] Classify distributions u on $\mathbb{R}^2$ such that $r^2\cdot u = 0$, where r is radius.

[05.14] Show that $u \in H^{-1}(\mathbb{R})$ is $f'' - f$ for some continuous function f.

[05.15] Show that the (distributional) derivative of a *finite* positive, regular Borel measure μ on $\mathbb{T}$ is in $H^{-\frac{3}{2}-\varepsilon}(\mathbb{T})$ for every $\varepsilon > 0$. (Hint: Riesz-Markov-Kakutani theorem.)

[05.16] Prove the *Sokhotski-Plemelj formula* (often arising in physics settings): for Schwartz f,

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \frac{f(x)\,dx}{x\pm i\varepsilon} \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} \frac{f(x)\,dx}{x} \;\mp\; \pi i\cdot f(0)$$

(*Hint:* the earlier example of the limiting behavior of $\varepsilon/(\varepsilon^2+x^2)$ is helpful.)
