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Examples 06

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-06.pdf]

For feedback on these examples, please get your write-ups to me by Friday, Feb 28.

[06.1] On $\mathbb{T}$, show that $u'' = \delta_{\mathbb{Z}}$ has no solution $u \in \mathcal{D}^*$.

[06.2] Define *translation* $T_{x_o}u$ of a distribution u by an amount $x_o \in \mathbb{R}$ by

$$(T_{x_o}u)(\varphi) \;=\; u(T_{-x_o}\varphi \qquad\qquad (\text{for } \varphi \in \mathcal{D})$$

The sign is for compatibility with distributions arising as integrate-against test functions. For tempered u, express $\widehat{T_{x_o}u}$ in terms of $\widehat{u}$.

[06.3] Compute $\widehat{\cos x}$.

[06.4] On $\mathbb{R}^n$, show that $|x|^2 \cdot \Delta\delta = 2n \cdot \delta$.

[06.5] Compute the Fourier transform of the *sign* function

$$\mathrm{sgn}(x) \;=\; \begin{cases} -1 & (x < 0) \\ \\ +1 & (x > 0) \end{cases}$$

[06.6] Compute the two-dimensional Fourier transform of $(x \pm iy)^n \cdot e^{-\pi(x^2+y^2)}$. (*Hint:* It is useful to rewrite things in terms of a complex variable $z = x + iy$ and its complex conjugate $\overline{z}$.)

[06.7] The Cauchy-Riemann operator on $\mathbb{C} \approx \mathbb{R}^2$ is

$$\overline{\partial} \;=\; \frac{\partial}{\partial \overline{z}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

Let $u_{n,s}(z) = \big(\frac{z}{|z|}\big)^n \cdot |z|^s$ for $n \in \mathbb{Z}$ and $s \in \mathbb{C}$. Determine the requirements on n, s such that $u_{n,s}$ is locally integrable (and, thus, because it is of moderate growth, gives a tempered distributions). Compute $\overline{\partial}u_{n,s}$, and explain how to interpret the outcome in case the outcome is no longer locally integrable.

[06.8] On $\mathbb{R}^n$, for fixed $\varphi \in \mathcal{D}$, show that the function $f\varphi(s) = \int_{\mathbb{R}^n} \varphi(x)\,|x|^s\,dx$ blows up as $s \to -n^+$, in particular, there is a constant C_n such that

$$f\varphi(s) \;=\; \frac{C_n \cdot \varphi(0)}{s+n} + (\text{continuous at } -n)$$

(Thus, if we understand that $s \to$ integration-against $|x|^s$ is a *meromorphic* distribution-valued function, its *residue* at $s = -n$ is a constant multiple of δ.)

[06.9] The Riemann-equation characterizing holomorphic functions f is $\overline{\partial}f = 0$. Show that

$$\overline{\partial}\,\frac{1}{z} \;=\; (\text{constant multiple of})\,\delta$$

[06.10] On $\mathbb{R}^2 \approx \mathbb{C}$, show that $Tf(z) = f(z)/z\}$ is a continuous map of the subspace $\mathscr{S}_1 = \{f \in \mathscr{S}(\mathbb{R}^2) : f(\mu z) = \mu \cdot f(z)\, \forall |\mu| = 1\}$ to $C^o(\mathbb{R}^2)$. (*Hint:* Use Taylor-Maclaurin series.)

[06.11] On $\mathbb{R}^2 \approx \mathbb{C}$, show that the principal-value integral

$$u(f) \;=\; \lim_{\varepsilon \to 0^+} \int_{|z| \geq \varepsilon} f(z)\, \frac{z}{|z|^3}\, dx\, dy \qquad\qquad (\text{for } f \in \mathscr{S})$$

gives a tempered distribution.

[06.12] Compute the Fourier transform of the distribution in the previous example.
