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Examples 07

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-07.pdf]

For feedback on these examples, please get your write-ups to me by Friday, April 03, 2020.

[07.1] Recall the proof of the spectral theorem for self-adjoint operators on a finite-dimensional complex vector space V with hermitian inner product.

[07.2] Recall the proof of a spectral theorem for *two* self-adjoint operators S, T on a finite-dimensional complex vector space V under the assumption that $ST = TS$.

[07.3] Let $K(x, y) = |x - y|$, and let

$$Tf(x) = \int_a^b K(x, y) f(y) dy \quad (\text{for } f \in L^2[a, b])$$

Find some eigenvalues/eigenfunctions for the operator T . (*Hint*: consider $\frac{d^2}{dx^2}(Tf)$ and use the fundamental theorem of calculus.)

[07.4] Let $K(x, y) \in L^2([a, b] \times [a, b])$, and attempt to define a map $T : L^2[a, b] \rightarrow L^2[a, b]$ by

$$Tf(x) = \int_a^b K(x, y) f(y) dy$$

Show that Tf is well-defined a.e. as a pointwise-valued function. Show that T really does map L^2 to itself by showing that

$$\|Tf\|_{L^2[a, b]} \leq \|K\|_{L^2([a, b] \times [a, b])} \cdot \|f\|_{L^2[a, b]}$$

(One would say that $K(\cdot, \cdot)$ is a *Schwartz kernel* for the map T . Yes, this use is in conflict with the use of *kernel* of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint T^* of T has Schwartz kernel $\overline{K(y, x)}$. In fact, the map T is a *Hilbert-Schmidt* operator, and is therefore *compact*.)

[07.5] Prove that the *Volterra operator* $Vf(x) = \int_0^x f(t) dt$ on $C^0[0, 1]$ or on $L^2[0, 1]$ has no (not-identically-zero) eigenvalues/eigenvectors (despite being *compact*!)

[07.6] Determine the spectrum of the *left-shift* $L : (c_1, c_2, \dots) \rightarrow (c_2, \dots)$ on ℓ^2 , and of the *right-shift* $R : (c_1, c_2, \dots) \rightarrow (0, c_1, c_2, \dots)$ on ℓ^2 . Show that these are mutual *adjoints*.

[07.7] (*Approximate eigenvectors and continuous spectrum, Weyl's criterion*) Let $T : V \rightarrow V$ be a self-adjoint linear operator on a Hilbert space V . For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) such that $(T - \lambda)v_n \rightarrow 0$ as $n \rightarrow +\infty$ is an *approximate eigenvector* for λ . Show that for λ *not* an eigenvalue for T , λ has an approximate eigenvector if and only if λ is in the spectrum of T .

[07.8] Show that the multiplication operator $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by $Tf(x) = f(x) \cdot \sin x$ has empty discrete spectrum. Show that it is self-adjoint. Show that T has continuous spectrum the interval $[-1, 1]$. (We know that self-adjoint (or merely *normal*) operators have only point spectrum and continuous spectrum, that is, no left-over *residual* spectrum.)

[07.9] Let r_1, r_2, r_3, \dots be an enumeration of the rational numbers inside the interval $[0, 1]$. Define $T : \ell^2 \rightarrow \ell^2$ by $T(c_1, c_2, \dots) = (r_1 c_1, r_2 c_2, \dots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the r_1, r_2, \dots , and continuous spectrum the whole interval $[0, 1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

[07.10] Let r_1, r_2, r_3, \dots be a bounded sequence of complex numbers. Define $T : \ell^2 \rightarrow \ell^2$ by $T(c_1, c_2, \dots) = (r_1 c_1, r_2 c_2, \dots)$. Show that T is *compact* if and only if $r_n \rightarrow 0$.

[07.11] Let T be a compact operator $T : V \rightarrow W$ for Hilbert spaces V, W . For S a continuous/bounded operator on V , show that $T \circ S : V \rightarrow W$ is compact. For R a continuous/bounded operator on W , show that $R \circ T : V \rightarrow W$ is compact.

[07.12] Let S, T be two compact, self-adjoint operators on a Hilbert space, and $ST = TS$. Show that there is an orthonormal basis for V consisting of simultaneous eigenfunctions for S, T .

[07.13] Let r_1, r_2, r_3, \dots be a bounded sequence of complex numbers. Define $T : \ell^2 \rightarrow \ell^2$ by $T(c_1, c_2, \dots) = (r_1 c_1, r_2 c_2, \dots)$. Show that T is *Hilbert-Schmidt* if and only if $\sum |r_n|^2 < \infty$.

[07.14] Let r_1, r_2, r_3, \dots be a bounded sequence of complex numbers. Define $T : \ell^2 \rightarrow \ell^2$ by $T(c_1, c_2, \dots) = (r_1 c_1, r_2 c_2, \dots)$. Show that T is *trace class* if and only if $\sum |r_n| < \infty$.
