(*April 25, 2020*)

Examples 08

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-08.pdf]

For feedback on these examples, please get your write-ups to me by Friday, April 24, 2020.

[08.1] For abelian groups A, B, C, prove that $\mathrm{Hom}(A \otimes B, C) \approx \mathrm{Hom}(A, \mathrm{Hom}(B, C))$.

[08.2] For a field k and a k-vectorspace V (without topology), show that the map $V \otimes V^* \to \mathrm{End}_k V$ induced from $(v \otimes \lambda)(w) = \lambda(w) \cdot v$, for $v, w \in V$ and $\lambda \in V^*$, is a bijection to *finite-rank* endomorphisms of V (meaning that their images are finite-dimensional).

[08.3] *(Coordinate-independent expression for trace)* In the situation of the previous example, let $\Phi_k(V)$ be the finite-rank endomorphisms of V, and $T : \Phi_k(V) \to V \otimes V^*$ the inverse of the map given there. Let $\beta : V \times V^*$ be the bilinear k-valued map $v \times \lambda \to \lambda(v)$, inducing a linear k-valued map $B : V \otimes V^*$ given by $B(v \otimes \lambda) = \lambda(v)$. Show that $B \otimes T : \Phi_k(B) \to k$ is the *trace* map on finite-rank endomorphisms.

[08.4] Show that there is no continuous extension of *trace* from finite-rank operators on an infinite-dimensional Hilbert space to all continuous operators, and not even to all Hilbert-Schmidt operators.

[08.5] Determine in which Sobolev space(s) $H^s(\mathbb{T}^2)$ the Schwartz kernel for the inclusion
$T : \mathcal{D}(\mathbb{T}) \to test(\mathbb{T})^*$ lies.

[08.6] Determine in which Sobolev space(s) $H^s(\mathbb{T}^2)$ the Schwartz kernel for the differentiation map
$\frac{d}{dx} : \mathcal{D}(\mathbb{T}) \to \mathcal{D}(\mathbb{T})^*$ lies.

[08.7] Determine the Schwartz kernel for the Fourier-Plancherel transform $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

[08.8] Show that an operator $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by a Schwartz kernel $K(x, y)$ in $H^{\frac{1}{2}+\varepsilon}(\mathbb{T} \times \mathbb{T})$ is trace-class. Show that $x \to K(x, x)$ is in $L^2(\mathbb{T})$. Show that T has trace $\mathrm{tr}\, T = \int_{\mathbb{T}} K(x, x)\, dx$. (*Hint:* Trace theorem... with different sense of *trace*.)

[08.9] Let

$$K(x, y) \;=\; \begin{cases} x \cdot (\frac{y}{2\pi} - 1) & (\text{in } 0 < x < y) \\ \\ y \cdot (\frac{x}{2\pi} - 1) & (\text{in } 2\pi > x > y) \end{cases}$$

be the kernel for a continuous linear map $T : L^2[0, 2\pi] \to L^2[0, 2\pi]$. We have seen that T is *compact* and self-adjoint, with eigenvectors $\sin \frac{nx}{2}$, for $n = 1, 2, 3, \ldots$. Show that T is trace-class. Take its trace to give yet another proof that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \quad = \quad \frac{\pi^2}{6}$$
