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Examples 01

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2022-23/real-ex-01.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 26 Sept 2022.

[01.1] Let $c_1 > c_2 > c_3 > \dots > 0$ with $c_n \rightarrow 0$. Show that, for all $0 < x < 2\pi$, $\sum_n c_n e^{inx}$ is *convergent*.

[01.2] Let f_n be the piece-wise linear tent functions with heights n , widths $2/n$, centered at $1/n$: formulaically,

$$f_n(x) = \begin{cases} 0 & (\text{for } x \leq 0 \text{ or } x \geq \frac{2}{n}) \\ n^2 \cdot x & (\text{for } 0 \leq x \leq \frac{1}{n}) \\ n - n^2(x - \frac{1}{n}) & (\text{for } \frac{1}{n} \leq x \leq \frac{2}{n}) \end{cases}$$

Show that for $g \in C^o(\mathbb{R})$,

$$\lim_n \int_{\mathbb{R}} f_n(x) g(x) dx = g(0)$$

[01.3] There is not much hope in making sense of the outcome of an uncountable number of non-zero operations: let Ω be an *uncountable* collection of positive real numbers. Letting F range over all finite subsets of Ω , show that $\sup_F \sum_{\alpha \in F} \alpha = +\infty$.

[01.4] Show that the closed unit ball in ℓ^2 is *not compact*.

[01.5] Show that the closed unit ball in $C^o[a, b]$ is not compact.

[01.6] Show that $C^o[a, b]$ is *not complete* with the $L^2[a, b]$ metric.

[01.7] Show that $C^1[a, b]$ is *not complete* with the $C^o[a, b]$ metric.

[01.8] Show that $C^1[a, b]$ is complete, with the $C^1[a, b]$ metric

$$d(f, g) = \sup_{a \leq x \leq b} |f(x) - g(x)| + \sup_{a \leq x \leq b} |f'(x) - g'(x)|$$

[01.9] Show that the *Hilbert cube*

$$C = \{(z_1, z_2, \dots) \in \ell^2 : |z_n| \leq \frac{1}{n}\}$$

is compact. (Hint: use the *total boundedness* criterion.)

[01.10] The space of continuous functions on \mathbb{R} *going to 0 at infinity* is

$$C_o^o(\mathbb{R}) = \{f \in C^o(\mathbb{R}) : \text{for every } \varepsilon > 0 \text{ there is } T \text{ such that } |f(x)| < \varepsilon \text{ for all } |x| \geq T\}$$

Show that the closure of $C_c^o(\mathbb{R})$ in the space $C_{\text{bdd}}^o(\mathbb{R})$ of bounded continuous functions with sup norm, is $C_o^o(\mathbb{R})$.