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Examples 02

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/examples_2022-23/real-ex-02.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 17 Oct 2022.

[02.1] Show that every open subset of $\mathbb{R}^n$ is a *countable* union of open balls.

[02.2] For positive real $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\le\; w_1 a_1 + \ldots + w_n a_n$$

[02.3] *Lebesgue (outer) measure* $\mu(E)$ of subsets E of $\mathbb{R}$ is

$$\mu(E) \;=\; \inf\{\sum_{n=1}^\infty |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^\infty (a_n, b_n)\}$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where M is Cantor's middle-thirds set.

[02.4] Show that for measurable f on $[a, b]$,

$$\Big|\int_a^b f(x)\, dx\Big|^2 \;\le\; |b-a| \cdot \int_a^b |f(x)|^2\, dx$$

with equality only for f (almost-everywhere) constant.

[02.5] For non-negative, real-valued f, show that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(x)\, e^{-\varepsilon x^2}\, dx \;=\; \int_{\mathbb{R}} f(x)\, dx$$

[02.6] For $g \in C_o^o(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, show that

$$\lim_{t \to +\infty} \int_{\mathbb{R}} f(x)\, g(x+t)\, dx \;=\; 0$$

[02.7] Functions in $L^1(\mathbb{R})$ need not go to 0 at infinity: give an example of $f \in L^1(\mathbb{R})$ such that $\limsup_{x\to+\infty} |f(x)| = +\infty$.

[02.8] For $f \in L^1(\mathbb{R})$, show that

$$\lim_{\varepsilon \to 0^+} \int_0^\varepsilon f(x)\, dx \;=\; 0$$

[02.9] For $f \in L^2(\mathbb{R})$, show that there is a constant C such that

$$\Big|\int_0^\varepsilon f(x)\, dx\Big| \;\le\; C \cdot \sqrt{\varepsilon}$$

for $0 < \varepsilon \le 1$.

[02.10] Let f be a continuous function on $[0, 1]$, with $f(0) = 0$ and $f(1) = 1$. Show that the set $\{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\}$ has positive measure.

[02.11] Show that $\ell^p \subset \ell^q$ for $1 < p < q < \infty$, and that the containment is *proper*.
