Examples 03

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2022-23/real-ex-03.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 07 Nov 2022.

03.1 For a vector subspace \( W \) of a Hilbert space \( V \), show that \((W^\perp)^\perp\) is the topological closure of \( W \).

03.2 Find two dense vector subspaces \( X, Y \) of \( \ell^2 \) such that \( X \cap Y = \{0\} \). (And, if you need further entertainment, can you find countably-many dense vector subspaces \( X_n \) such that \( X_m \cap X_n = \{0\} \) for \( m \neq n \)?)

03.3 For measurable \( E \subset [0,1] \), show that \( \lim_n \int_E e^{-2\pi inx} \, dx = 0 \) as \( n \to \infty \) ranging over integers.

03.4 Let \( f_n(x) = \sin \pi nx \) on \([0,1]\), extended by \( \mathbb{Z} \)-periodicity, for \( n = 1, 2, 3, \ldots \). Given \( g \in L^1[0,1] \), show that \( \int_0^1 f_n \cdot g \to 0 \).

03.5 Compute the Fourier coefficients of the sawtooth function \( s(x) = x - \frac{1}{2} \) on \([0,1]\), extended by \( \mathbb{Z} \)-periodicity. Use this to show that \( \sum_{n \geq 1} 1/n^2 = \pi^2/6 \).

03.6 Let \( E \) be a Lebesgue measurable set in \( \mathbb{R} \) with finite Lebesgue measure. Show that

\[
\lim_{t \to +\infty} \int_E \sin tx \, dx = 0 \quad \text{(over real \( t \))}
\]

03.7 Compute \( \int_{\mathbb{R}} \left( \frac{\sin x}{x} \right)^2 \, dx \). (Hint: do not attempt to do this directly, nor by complex analysis.)

03.8 (Collecting Fourier transform pairs) Compute the Fourier transforms of

\[
\chi_{[a,b]} e^{-\pi x^2} \quad f(x) = \begin{cases} e^{-x} & \text{(for } x > 0) \\ 0 & \text{(for } x \leq 0) \end{cases}
\]

03.9 Give an explicit non-zero function \( f \) such that \( \int_{\mathbb{R}} x^n f(x) \, dx = 0 \), for all \( n = 0, 1, 2, \ldots \).

03.10 Show that \( \chi_{[a,b]} \ast \chi_{[c,d]} \) is a piecewise-linear function, and express it explicitly.

03.11 For \( f \in \mathcal{S} \), show that

\[
\lim_{\varepsilon \to 0^+} f(x) \ast \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} = f(x)
\]

03.12 (Corrected!) For \( f \in \mathcal{S} \), show that

\[
\lim_{t \to +\infty} f(x) \ast \frac{\sin 2\pi tx}{\pi x} = f(x)
\]
Evaluate the Borwein integral

\[ \int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \, dx \]