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Examples 04

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2022-23/real-ex-04.pdf]

For feedback on these examples, please get your write-ups to me by Wed, 05 Dec 2022.

[04.1] Give a *persuasive* proof that the function

$$f(x) = \begin{cases} 0 & (\text{for } x \leq 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this to make a *smooth step function*: 0 for $x \leq 0$ and 1 for $x \geq 1$, and goes monotonically from 0 to 1 in the interval $[0, 1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1, 2, 3, \dots\}$: for each n , $f_n(x) = 1$ for $x \in [-n, n]$, $f_n(x) = 0$ for $x \notin [-(n+1), n+1]$, and f_n goes monotonically from 0 to 1 in $[-(n+1), -n]$ and monotonically from 1 to 0 in $[n, n+1]$.

[04.2] Use a family of smooth cut-offs to show that test functions on \mathbb{R} are dense in Schwartz functions on \mathbb{R} .

[04.3] Show that multiplication by x , and also differentiation d/dx , are continuous linear maps $\mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$.

[04.4] Show that $\delta(\varphi) = \varphi(0)$ is a tempered distribution.

[04.5] Show that the principal value integral $\lim_{\varepsilon \rightarrow 0^+} \int_{|x| > \varepsilon} \frac{f(x)}{x} dx$ is a tempered distribution, and satisfies $x \cdot u = 1$.

[04.6] Reprove $\widehat{\delta} = 1$ by approximating δ by Gaussians.

[04.7] Show that $\lim_n \frac{1}{1 + (x - n)^2} = 0$ in $\mathcal{S}(\mathbb{R})^*$.

[04.8] Determine the constant c such that $x^2 \delta'' = c \cdot \delta$.

[04.9] Show that $\sin(nx) \rightarrow 0$ in the \mathcal{S}^* -topology as $n \rightarrow +\infty$.

[04.10] Solve $u'' - \lambda \cdot u = \delta$ on \mathbb{R} , for $\lambda \notin \mathbb{R}$.
