Examples 07

For feedback on these examples, please get your write-ups to me by Monday, April 03, 2023.

[07.1] Recall the proof of the spectral theorem for self-adjoint operators on a finite-dimensional complex vector space $V$ with hermitian inner product.

[07.2] Recall the proof of a spectral theorem for two self-adjoint operators $S, T$ on a finite-dimensional complex vector space $V$ under the assumption that $ST = TS$.

[07.3] Let $K(x, y) = |x - y|$, and let

$$Tf(x) = \int_a^b K(x, y) f(y) \, dy \quad \text{(for } f \in L^2[a, b])$$

Find some eigenvalues/eigenfunctions for the operator $T$. (Hint: consider $\frac{d^2}{dx^2}(Tf)$ and use the fundamental theorem of calculus.)

[07.4] Let $K(x, y) \in L^2([a, b] \times [a, b])$, and attempt to define a map $T : L^2[a, b] \to L^2[a, b]$ by

$$Tf(x) = \int_a^b K(x, y) f(y) \, dy$$

Show that $Tf$ is well-defined a.e. as a pointwise-valued function. Show that $T$ really does map $L^2$ to itself by showing that

$$|Tf|_{L^2[a, b]} \leq |K|_{L^2([a, b] \times [a, b])} \cdot |f|_{L^2[a, b]}$$

(One would say that $K(\cdot, \cdot)$ is a Schwartz kernel for the map $T$. Yes, this use is in conflict with the use of kernel of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint $T^*$ of $T$ has Schwartz kernel $K(y, x)$. In fact, the map $T$ is a Hilbert-Schmidt operator, and is therefore compact.

[07.5] Prove that the Volterra operator $Vf(x) = \int_0^x f(t) \, dt$ on $C^0[0, 1]$ or on $L^2[0, 1]$ has no (not-identically-zero) eigenvalues/eigenfunctions (despite being compact!)

[07.6] Determine the spectrum of the left-shift $L : (c_1, c_2, \ldots) \to (c_2, \ldots)$ on $\ell^2$, and of the right-shift $R : (c_1, c_2, \ldots) \to (0, c_1, c_2, \ldots)$ on $\ell^2$. Show that these are mutual adjoints.

[07.7] (Approximate eigenvectors and continuous spectrum, Weyl’s criterion) Let $T : V \to V$ be a self-adjoint linear operator on a Hilbert space $V$. For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) such that $(T - \lambda)v_n \to 0$ as $n \to +\infty$ is an approximate eigenvector for $\lambda$. Show that for $\lambda$ not an eigenvalue for $T$, $\lambda$ has an approximate eigenvector if and only if $\lambda$ is in the spectrum of $T$.

[07.8] Show that the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Tf(x) = f(x) \cdot \sin x$ has empty discrete spectrum. Show that it is self-adjoint. Show that $T$ has continuous spectrum the interval $[-1, 1]$. (We know that self-adjoint (or merely normal) operators have only point spectrum and continuous spectrum, that is, no left-over residual spectrum.)
[07.9] Let \( r_1, r_2, r_3, \ldots \) be an enumeration of the rational numbers inside the interval \([0, 1]\). Define \( T : \ell^2 \to \ell^2 \) by \( T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \). Show that \( T \) is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the \( r_1, r_2, \ldots \), and continuous spectrum the whole interval \([0, 1]\) (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

[07.10] Let \( r_1, r_2, r_3, \ldots \) be a bounded sequence of complex numbers. Define \( T : \ell^2 \to \ell^2 \) by \( T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \). Show that \( T \) is compact if and only if \( r_n \to 0 \).

[07.11] Let \( T \) be a compact operator \( T : V \to W \) for Hilbert spaces \( V, W \). For \( S \) a continuous/bounded operator on \( V \), show that \( T \circ S : V \to W \) is compact. For \( R \) a continuous/bounded operator on \( W \), show that \( R \circ T : V \to W \) is compact.

[07.12] Let \( S, T \) be two compact, self-adjoint operators on a Hilbert space, and \( ST = TS \). Show that there is an orthonormal basis for \( V \) consisting of simultaneous eigenfunctions for \( S, T \).

[07.13] Let \( r_1, r_2, r_3, \ldots \) be a bounded sequence of complex numbers. Define \( T : \ell^2 \to \ell^2 \) by \( T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \). Show that \( T \) is Hilbert-Schmidt if and only if \( \sum |r_n|^2 < \infty \).

[07.14] Let \( r_1, r_2, r_3, \ldots \) be a bounded sequence of complex numbers. Define \( T : \ell^2 \to \ell^2 \) by \( T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \). Show that \( T \) is trace class if and only if \( \sum |r_n| < \infty \).