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Banach spaces $C^k[a, b]$

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1. Banach spaces $C^k[a, b]$
2. Non-Banach limit $C^\infty[a, b]$ of Banach spaces $C^k[a, b]$

We specify natural topologies, in which differentiation or other natural operators are *continuous*, and so that the space is *complete*.

Many familiar and useful spaces of continuous or differentiable functions, such as $C^k[a, b]$, have natural metric structures, and are *complete*. In these cases, the metric $d(\cdot, \cdot)$ comes from a *norm* $|\cdot|$, on the functions, giving Banach spaces.

Other natural function spaces, such as $C^\infty[a, b]$, are *not* Banach, but still do have a metric topology and are complete: these are *Fréchet spaces*, appearing as (projective) *limits* of Banach spaces, as below. These lack some of the conveniences of Banach spaces, but their expressions as *limits* of Banach spaces is often sufficient.

1. Banach spaces $C^k[a, b]$

We give the vector space $C^k[a, b]$ of k -times continuously differentiable functions on an interval $[a, b]$ a metric which makes it *complete*. Mere *pointwise* limits of continuous functions easily fail to be continuous. First recall the standard

[1.0.1] **Claim:** The set $C^0(K)$ of complex-valued continuous functions on a compact set K is *complete* with the metric $|f - g|_{C^0}$, with the C^0 -norm $|f|_{C^0} = \sup_{x \in K} |f(x)|$.

Proof: This is a typical three-epsilon argument. To show that a Cauchy sequence $\{f_i\}$ of continuous functions has a *pointwise* limit which is a continuous function, first argue that f_i has a pointwise limit at every $x \in K$. Given $\varepsilon > 0$, choose N large enough such that $|f_i - f_j| < \varepsilon$ for all $i, j \geq N$. Then $|f_i(x) - f_j(x)| < \varepsilon$ for any x in K . Thus, the sequence of values $f_i(x)$ is a Cauchy sequence of complex numbers, so has a limit $f(x)$. Further, given $\varepsilon' > 0$ choose $j \geq N$ sufficiently large such that $|f_j(x) - f(x)| < \varepsilon'$. For $i \geq N$

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| < \varepsilon + \varepsilon'$$

This is true for every positive ε' , so $|f_i(x) - f(x)| \leq \varepsilon$ for every x in K . That is, the pointwise limit is approached uniformly in $x \in [a, b]$.

To prove that $f(x)$ is continuous, for $\varepsilon > 0$, take N be large enough so that $|f_i - f_j| < \varepsilon$ for all $i, j \geq N$. From the previous paragraph $|f_i(x) - f(x)| \leq \varepsilon$ for every x and for $i \geq N$. Fix $i \geq N$ and $x \in K$, and choose a small enough neighborhood U of x such that $|f_i(x) - f_i(y)| < \varepsilon$ for any y in U . Then

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \leq \varepsilon + |f_i(x) - f_i(y)| + \varepsilon < \varepsilon + \varepsilon + \varepsilon$$

Thus, the pointwise limit f is continuous at every x in U . ///

Unsurprisingly, but significantly:

[1.0.2] **Claim:** For $x \in [a, b]$, the *evaluation* map $f \rightarrow f(x)$ is a continuous linear functional on $C^0[a, b]$.

Proof: For $|f - g|_{C^0} < \varepsilon$, we have

$$|f(x) - g(x)| \leq |f - g|_{C^0} < \varepsilon$$

proving the continuity. ///

As usual, a real-valued or complex-valued function f on a closed interval $[a, b] \subset \mathbb{R}$ is *continuously differentiable* when it has a derivative which is itself a continuous function. That is, the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists for all $x \in [a, b]$, and the function $f'(x)$ is in $C^0[a, b]$. Let $C^k[a, b]$ be the collection of k -times continuously differentiable functions on $[a, b]$, with the C^k -norm

$$\|f\|_{C^k} = \sum_{0 \leq i \leq k} \sup_{x \in [a, b]} |f^{(i)}(x)| = \sum_{0 \leq i \leq k} \|f^{(i)}\|_{\infty}$$

where $f^{(i)}$ is the i^{th} derivative of f . The associated metric on $C^k[a, b]$ is $\|f - g\|_{C^k}$.

Similar to the assertion about evaluation on $C^0[a, b]$,

[1.0.3] Claim: For $x \in [a, b]$ and $0 \leq j \leq k$, the evaluation map $f \rightarrow f^{(j)}(x)$ is a continuous linear functional on $C^k[a, b]$.

Proof: For $\|f - g\|_{C^k} < \varepsilon$,

$$|f^{(j)}(x) - g^{(j)}(x)| \leq \|f - g\|_{C^k} < \varepsilon$$

proving the continuity. ///

We see that $C^k[a, b]$ is a Banach space:

[1.0.4] Theorem: The normed metric space $C^k[a, b]$ is complete.

Proof: For a Cauchy sequence $\{f_i\}$ in $C^k[a, b]$, all the pointwise limits $\lim_i f_i^{(j)}(x)$ of j -fold derivatives exist for $0 \leq j \leq k$, and are uniformly continuous. The issue is to show that $\lim_i f_i^{(j)}$ is differentiable, with derivative $\lim_i f_i^{(j+1)}$. It suffices to show that, for a Cauchy sequence f_n in $C^1[a, b]$, with pointwise limits $f(x) = \lim_n f_n(x)$ and $g(x) = \lim_n f_n'(x)$ we have $g = f'$. By the fundamental theorem of calculus, for any index i ,

$$f_i(x) - f_i(a) = \int_a^x f_i'(t) dt$$

Since the f_i' uniformly approach g , given $\varepsilon > 0$ there is i_0 such that $|f_i'(t) - g(t)| < \varepsilon$ for $i \geq i_0$ and for all t in the interval, so for such i

$$\left| \int_a^x f_i'(t) dt - \int_a^x g(t) dt \right| \leq \int_a^x |f_i'(t) - g(t)| dt \leq \varepsilon \cdot |x - a| \rightarrow 0$$

Thus,

$$\lim_i f_i(x) - f_i(a) = \lim_i \int_a^x f_i'(t) dt = \int_a^x g(t) dt$$

from which $f' = g$. ///

By design, we have

[1.0.5] Theorem: The map $\frac{d}{dx} : C^k[a, b] \rightarrow C^{k-1}[a, b]$ is continuous.

Proof: As usual, for a linear map $T : V \rightarrow W$, by linearity $Tv - Tv' = T(v - v')$ it suffices to check continuity at 0. For Banach spaces the homogeneity $|\sigma \cdot v|_V = |\alpha| \cdot |v|_V$ shows that continuity is equivalent to existence of a constant B such that $|Tv|_W \leq B \cdot |v|_V$ for $v \in V$. Then

$$\left\| \frac{d}{dx} f \right\|_{C^{k-1}} = \sum_{0 \leq i \leq k-1} \sup_{x \in [a, b]} \left| \left(\frac{df}{dx} \right)^{(i)}(x) \right| = \sum_{1 \leq i \leq k} \sup_{x \in [a, b]} |f^{(i)}(x)| \leq 1 \cdot \|f\|_{C^k}$$

as desired. ///

2. Non-Banach limit $C^\infty[a, b]$ of Banach spaces $C^k[a, b]$

The space $C^\infty[a, b]$ of infinitely differentiable complex-valued functions on a (finite) interval $[a, b]$ in \mathbb{R} is not a Banach space. ^[1] Nevertheless, the topology is *completely determined* by its relation to the Banach spaces $C^k[a, b]$. That is, there is a *unique* reasonable topology on $C^\infty[a, b]$. After explaining and proving this uniqueness, we also show that this topology is *complete metric*.

This function space can be presented as

$$C^\infty[a, b] = \bigcap_{k \geq 0} C^k[a, b]$$

and we reasonably require that whatever topology $C^\infty[a, b]$ should have, each inclusion $C^\infty[a, b] \rightarrow C^k[a, b]$ is continuous.

At the same time, given a family of *continuous linear* maps $Z \rightarrow C^k[a, b]$ from a vector space Z in some reasonable class, with the *compatibility* condition of giving commutative diagrams

$$\begin{array}{ccc} C^k[a, b] & \xrightarrow{\subset} & C^{k-1}[a, b] \\ & \swarrow & \uparrow \\ & & Z \end{array}$$

the image of Z actually lies in the intersection $C^\infty[a, b]$. Thus, diagrammatically, for every family of compatible maps $Z \rightarrow C^k[a, b]$, there is a *unique* $Z \rightarrow C^\infty[a, b]$ fitting into a commutative diagram

$$\begin{array}{ccccc} & & \curvearrowright & & \\ C^\infty[a, b] & \xrightarrow{\quad} & C^1[a, b] & \xrightarrow{\quad} & C^0[a, b] \\ & \swarrow \exists! & \nearrow \forall & \searrow & \\ & & Z & & \end{array}$$

We require that this induced map $Z \rightarrow C^\infty[a, b]$ is *continuous*.

When we know that these conditions are met, we would say that $C^\infty[a, b]$ is the (projective) *limit* of the spaces $C^k[a, b]$, written

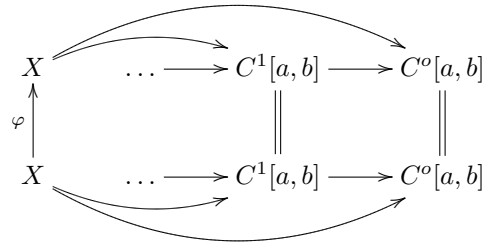
$$C^\infty[a, b] = \lim_k C^k[a, b]$$

with implicit reference to the inclusions $C^{k+1}[a, b] \rightarrow C^k[a, b]$ and $C^\infty[a, b] \rightarrow C^k[a, b]$.

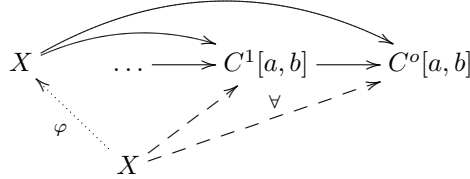
[2.0.1] Claim: Up to unique isomorphism, there exists at most one topology on $C^\infty[a, b]$ such that to every compatible family of continuous linear maps $Z \rightarrow C^k[a, b]$ from a topological vector space Z there is a unique continuous linear $Z \rightarrow C^\infty[a, b]$ fitting into a commutative diagram as just above.

Proof: Let X, Y be $C^\infty[a, b]$ with two topologies fitting into such diagrams, and show $X \approx Y$, and for a unique isomorphism. First, claim that the identity map $\text{id}_X : X \rightarrow X$ is the only map $\varphi : X \rightarrow X$ fitting into a commutative diagram

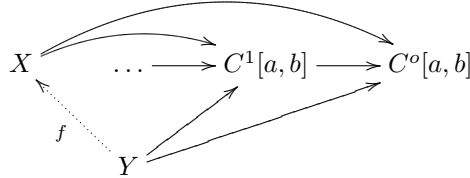
^[1] It is not essential to prove that there is no reasonable Banach space structure on $C^\infty[a, b]$, but this can be readily proven in a suitable context.



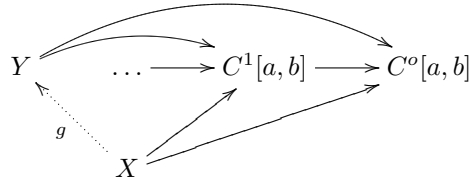
Indeed, given a compatible family of maps $X \rightarrow C^k[a, b]$, there is *unique* φ fitting into



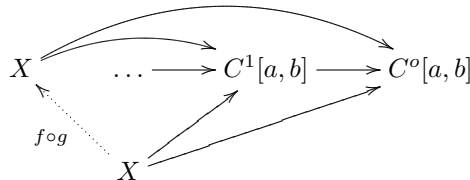
Since the identity map id_X fits, necessarily $\varphi = \text{id}_X$. Similarly, given the compatible family of inclusions $Y \rightarrow C^k[a, b]$, there is unique $f : Y \rightarrow X$ fitting into



Similarly, given the compatible family of inclusions $X \rightarrow C^k[a, b]$, there is unique $g : X \rightarrow Y$ fitting into



Then $f \circ g : X \rightarrow X$ fits into a diagram

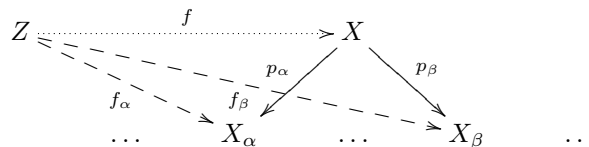


Therefore, $f \circ g = \text{id}_X$. Similarly, $g \circ f = \text{id}_Y$. That is, f, g are mutual inverses, so are isomorphisms of topological vector spaces. ///

Existence of a topology on $C^\infty[a, b]$ satisfying the condition above will be proven by identifying $C^\infty[a, b]$ as the obvious diagonal *closed subspace* of the *topological product* of the *limitands* $C^k[a, b]$:

$$C^\infty[a, b] = \{ \{f_k : f_k \in C^k[a, b]\} : f_k = f_{k+1} \text{ for all } k \}$$

An arbitrary *product* of topological spaces X_α for α in an index set A is a topological space X with (*projections*) $p_\alpha : X \rightarrow X_\alpha$, such that every family $f_\alpha : Z \rightarrow X_\alpha$ of maps from any other topological space Z *factors through* the p_α *uniquely*, in the sense that there is a unique $f : Z \rightarrow X$ such that $f_\alpha = p_\alpha \circ f$ for all α . Pictorially, *all triangles commute* in the diagram



A similar argument to that for uniqueness of limits proves *uniqueness* of products up to unique isomorphism. *Construction* of products is by putting the usual product topology with basis consisting of products $\prod_{\alpha} Y_{\alpha}$ with $Y_{\alpha} = X_{\alpha}$ for all but finitely-many indices, on the Cartesian product of the *sets* X_{α} , whose existence we grant ourselves. Proof that this usual is a product amounts to unwinding the definitions. By uniqueness, in particular, despite the plausibility of the *box topology* on the product, it cannot function as a product topology since it differs from the standard product topology in general.

[2.0.2] **Claim:** Giving the diagonal copy of $C^{\infty}[a, b]$ inside $\prod_k C^k[a, b]$ the subspace topology yields a (projective) limit topology.

Proof: The projection maps $p_k : \prod_j C^j[a, b] \rightarrow C^k[a, b]$ from the whole product to the factors $C^k[a, b]$ are continuous, so their restrictions to the diagonally imbedded $C^{\infty}[a, b]$ are continuous. Further, letting $i_k : C^k[a, b] \rightarrow C^{k-1}[a, b]$ be the inclusion, on that diagonal copy of $C^{\infty}[a, b]$ we have $i_k \circ p_k = p_{k-1}$ as required.

On the other hand, *any* family of maps $\varphi_k : Z \rightarrow C^k[a, b]$ induces a map $\tilde{\varphi} : Z \rightarrow \prod C^k[a, b]$ such that $p_k \circ \tilde{\varphi} = \varphi_k$, by the property of the product. *Compatibility* $i_k \circ \varphi_k = \varphi_{k-1}$ implies that the image of $\tilde{\varphi}$ is inside the diagonal, that is, inside the copy of $C^{\infty}[a, b]$. ///

A *countable* product of *metric* spaces X_k with metrics d_k has no canonical single metric, but is *metrizable*. One of many topologically equivalent metrics is the usual

$$d(\{x_k\}, \{y_k\}) = \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k - y_k)}{d_k(x_k - y_k) + 1}$$

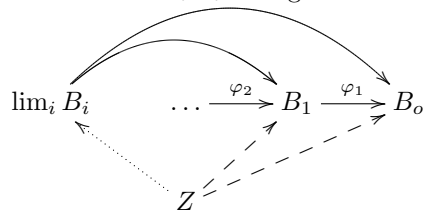
When the metric spaces X_k are *complete*, the product is complete. A closed subspace of a complete metrizable space is complete metrizable, so we have

[2.0.3] **Corollary:** $C^{\infty}[a, b]$ is complete metrizable. ///

Abstracting the above, for a (not necessarily countable) family

$$\dots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_0$$

of Banach spaces with continuous linear transition maps as indicated, *not* necessarily requiring the continuous linear maps to be injective (or surjective), a (*projective*) *limit* $\lim_i B_i$ is a topological vector space with continuous linear maps $\lim_i B_i \rightarrow B_j$ such that, for every compatible family of continuous linear maps $Z \rightarrow B_i$ there is unique continuous linear $Z \rightarrow \lim_i B_i$ fitting into



The same *uniqueness* proof as above shows that there is at most one topological vector space $\lim_i B_i$. For *existence* by *construction*, the earlier argument needs only minor adjustment. The conclusion of complete metrizability would hold when the family is countable.

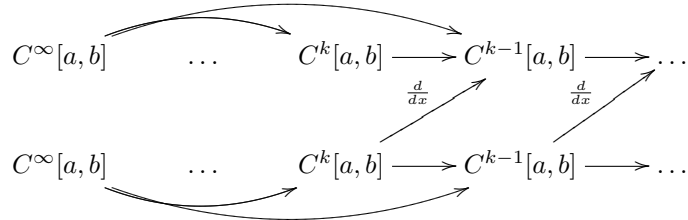
Before declaring $C^{\infty}[a, b]$ to be a *Fréchet* space, we must certify that it is *locally convex*, in the sense that every point has a local basis of *convex* opens. Normed spaces are immediately locally convex, because open balls are convex: for $0 \leq t \leq 1$ and x, y in the ε -ball at 0 in a normed space,

$$|tx + (1 - t)y| \leq |tx| + |(1 - t)y| \leq t|x| + (1 - t)|y| < t \cdot \varepsilon + (1 - t) \cdot \varepsilon = \varepsilon$$

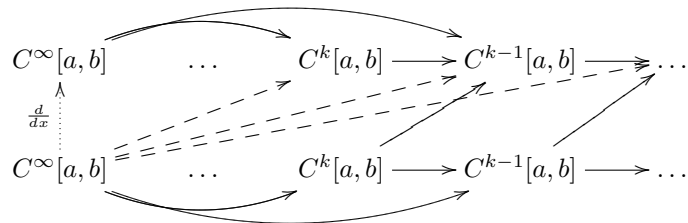
Product topologies of locally convex vectorspaces are locally convex, from the *construction* of the product. The construction of the limit as the diagonal in the product, with the subspace topology, shows that it is locally convex. In particular, *countable limits of Banach spaces are locally convex, hence, are Fréchet*. All spaces of practical interest are locally convex for simple reasons, so demonstrating local convexity is rarely interesting.

[2.0.4] **Theorem:** $\frac{d}{dx} : C^\infty[a, b] \rightarrow C^\infty[a, b]$ is continuous.

Proof: In fact, the differentiation operator is characterized via the expression of $C^\infty[a, b]$ as a limit. We already know that differentiation d/dx gives a continuous map $C^k[a, b] \rightarrow C^{k-1}[a, b]$. Differentiation is compatible with the inclusions among the $C^k[a, b]$. Thus, we have a commutative diagram



Composing the projections with d/dx gives (dashed) induced maps from $C^\infty[a, b]$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in



This proves the continuity of differentiation in the limit topology. ///

In a slightly different vein, we have

[2.0.5] **Claim:** For all $x \in [a, b]$ and for all non-negative integers k , the evaluation map $f \rightarrow f^{(k)}(x)$ is a continuous linear map $C^\infty[a, b] \rightarrow \mathbb{C}$.

Proof: The inclusion $C^\infty[a, b] \rightarrow C^k[a, b]$ is continuous, and the evaluation of the k^{th} derivative is continuous. ///