(*November 13, 2016*)

Banach spaces $C^k[a,b]$

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We specify natural topologies, in which differentiation or other natural operators are *continuous*, and so that the space is *complete*.

Many familiar and useful spaces of continuous or differentiable functions, such as $C^k[a,b]$, have natural metric structures, and are *complete*. In these cases, the metric $d(,)$ comes from a *norm* $|\cdot|$, on the functions, giving Banach spaces.

Other natural function spaces, such as $C^\infty[a,b]$, are *not* Banach, but still do have a metric topology and are complete: these are *Fréchet spaces*, appearing as (projective) *limits* of Banach spaces, as below. These lack some of the conveniences of Banach spaces, but their expressions as *limits* of Banach spaces is often sufficient.

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1. *Banach spaces* $C^k[a,b]$

We give the vector space $C^k[a,b]$ of k-times continuously differentiable functions on an interval $[a,b]$ a metric which makes it *complete*. Mere *pointwise* limits of continuous functions easily fail to be continuous. First recall the standard

[1.0.1] Claim: The set $C^o(K)$ of complex-valued continuous functions on a compact set K is *complete* with the metric $|f-g|_{C^o}$, with the C^o-*norm* $|f|_{C^o} = \sup_{x\in K} |f(x)|$.

Proof: This is a typical three-epsilon argument. To show that a Cauchy sequence $\{f_i\}$ of continuous functions has a *pointwise* limit which is a continuous function, first argue that f_i has a pointwise limit at every $x \in K$. Given $\varepsilon > 0$, choose N large enough such that $|f_i - f_j| < \varepsilon$ for all $i,j \ge N$. Then $|f_i(x) - f_j(x)| < \varepsilon$ for any x in K. Thus, the sequence of values $f_i(x)$ is a Cauchy sequence of complex numbers, so has a limit $f(x)$. Further, given $\varepsilon' > 0$ choose $j \ge N$ sufficiently large such that $|f_j(x) - f(x)| < \varepsilon'$. For $i \ge N$

$$|f_i(x) - f(x)| \;\le\; |f_i(x) - f_j(x)| + |f_j(x) - f(x)| \;<\; \varepsilon + \varepsilon'$$

This is true for every positive ε', so $|f_i(x) - f(x)| \le \varepsilon$ for every x in K. That is, the pointwise limit is approached uniformly in $x \in [a,b]$.

To prove that $f(x)$ is continuous, for $\varepsilon > 0$, take N be large enough so that $|f_i - f_j| < \varepsilon$ for all $i,j \ge N$. From the previous paragraph $|f_i(x) - f(x)| \le \varepsilon$ for every x and for $i \ge N$. Fix $i \ge N$ and $x \in K$, and choose a small enough neigborhood U of x such that $|f_i(x) - f_i(y)| < \varepsilon$ for any y in U. Then

$$|f(x) - f(y)| \;\le\; |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \;\le\; \varepsilon + |f_i(x) - f_i(y)| + \varepsilon \;<\; \varepsilon + \varepsilon + \varepsilon$$

Thus, the pointwise limit f is continuous at every x in U. ///

Unsurprisingly, but significantly:

[1.0.2] Claim: For $x \in [a,b]$, the *evaluation* map $f \to f(x)$ is a continuous linear functional on $C^o[a,b]$.

Proof: For $|f-g|_{C^o} < \varepsilon$, we have

$$|f(x) - g(x)| \;\le\; |f-g|_{C^o} \;<\; \varepsilon$$

proving the continuity. ///

As usual, a real-valued or complex-valued function f on a closed interval $[a,b] \subset \mathbb{R}$ is *continuously differentiable* when it has a derivative which is itself a continuous function. That is, the limit

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists for all $x \in [a,b]$, and the function $f'(x)$ is in $C^o[a,b]$. Let $C^k[a,b]$ be the collection of k-times continuously differentiable functions on $[a,b]$, with the C^k*-norm*

$$|f|_{C^k} \;=\; \sum_{0\le i\le k} \sup_{x\in[a,b]} |f^{(i)}(x)| \;=\; \sum_{0\le i\le k} |f^{(i)}|_\infty$$

where $f^{(i)}$ is the i^{th} derivative of f. The *associated metric* on $C^k[a,b]$ is $|f-g|_{C^k}$.

Similar to the assertion about evaluation on $C^o[a,b]$,

[1.0.3] Claim: For $x \in [a,b]$ and $0 \le j \le k$, the *evaluation* map $f \to f^{(j)}(x)$ is a continuous linear functional on $C^k[a,b]$.

Proof: For $|f-g|_{C^k} < \varepsilon$,

$$|f^{(j)}(x) - g^{(j)}(x)| \;\le\; |f-g|_{C^k} \;<\; \varepsilon$$

proving the continuity. ///

We see that $C^k[a,b]$ is a Banach space:

[1.0.4] Theorem: The normed metric space $C^k[a,b]$ is complete.

Proof: For a Cauchy sequence $\{f_i\}$ in $C^k[a,b]$, all the pointwise limits $\lim_i f_i^{(j)}(x)$ of j-fold derivatives exist for $0 \le j \le k$, and are uniformly continuous. The issue is to show that $\lim_i f^{(j)}$ is differentiable, with derivative $\lim_i f^{(j+1)}$. It suffices to show that, for a Cauchy sequence f_n in $C^1[a,b]$, with pointwise limits $f(x) = \lim_n f_n(x)$ and $g(x) = \lim_n f_n'(x)$ we have $g = f'$. By the fundamental theorem of calculus, for any index i,

$$f_i(x) - f_i(a) \;=\; \int_a^x f_i'(t)\,dt$$

Since the f_i' *uniformly* approach g, given $\varepsilon > 0$ there is i_o such that $|f_i'(t) - g(t)| < \varepsilon$ for $i \ge i_o$ and for *all* t in the interval, so for such i

$$\Big|\int_a^x f_i'(t)\,dt - \int_a^x g(t)\,dt\Big| \;\le\; \int_a^x |f_i'(t) - g(t)|\,dt \;\le\; \varepsilon\cdot|x-a| \;\longrightarrow\; 0$$

Thus,

$$\lim_i f_i(x) - f_i(a) \;=\; \lim_i \int_a^x f_i'(t)\,dt \;=\; \int_a^x g(t)\,dt$$

from which $f' = g$. ///

By design, we have

[1.0.5] Theorem: The map $\frac{d}{dx} : C^k[a,b] \to C^{k-1}[a,b]$ is continuous.

Proof: As usual, for a linear map $T : V \to W$, by linearity $Tv - Tv' = T(v-v')$ it suffices to check continuity at 0. For Banach spaces the homogeneity $|\sigma\cdot v|_V = |\alpha|\cdot|v|_V$ shows that continuity is equivalent to existence of a constant B such that $|Tv|_W \le B\cdot|v|_V$ for $v \in V$. Then

$$|\frac{d}{dx}f|_{C^{k-1}} \;=\; \sum_{0\le i\le k-1} \sup_{x\in[a,b]} |(\frac{df}{dx})^{(i)}(x)| \;=\; \sum_{1\le i\le k} \sup_{x\in[a,b]} |f^{(i)}(x)| \;\le\; 1\cdot|f|_{C^k}$$

as desired. ///

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2. Non-Banach limit $C^\infty[a,b]$ of Banach spaces $C^k[a,b]$

The space $C^\infty[a,b]$ of infinitely differentiable complex-valued functions on a (finite) interval $[a,b]$ in $\mathbb{R}$ is not a Banach space. [1] Nevertheless, the topology is *completely determined* by its relation to the Banach spaces $C^k[a,b]$. That is, there is a *unique* reasonable topology on $C^\infty[a,b]$. After explaining and proving this uniqueness, we also show that this topology is *complete metric*.

This function space can be presented as

$$C^\infty[a,b] \;=\; \bigcap_{k\ge 0} C^k[a,b]$$

and we reasonably require that whatever topology $C^\infty[a,b]$ should have, each inclusion $C^\infty[a,b] \longrightarrow C^k[a,b]$ is continuous.

At the same time, given a family of *continuous linear* maps $Z \to C^k[a,b]$ from a vector space Z in some reasonable class, with the *compatibility* condition of giving commutative diagrams

$$\begin{tikzcd}
C^k[a,b] \arrow[r, "\subset"] & C^{k-1}[a,b] \\
 & Z \arrow[ul] \arrow[u]
\end{tikzcd}$$

the image of Z actually lies in the intersection $C^\infty[a,b]$. Thus, diagrammatically, for every family of compatible maps $Z \to C^k[a,b]$, there is a *unique* $Z \to C^\infty[a,b]$ fitting into a commutative diagram

$$\begin{tikzcd}
C^\infty[a,b] \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
 & Z \arrow[ul, dotted, "\exists!"] \arrow[ur, dashed] \arrow[urr, dashed, "\forall"']
\end{tikzcd}$$

We require that this induced map $Z \to C^\infty[a,b]$ is *continuous*.

When we know that these conditions are met, we would say that $C^\infty[a,b]$ is the (projective) *limit* of the spaces $C^k[a,b]$, written

$$C^\infty[a,b] \;=\; \lim_k C^k[a,b]$$

with implicit reference to the inclusions $C^{k+1}[a,b] \to C^k[a,b]$ and $C^\infty[a,b] \to C^k[a,b]$.

[2.0.1] Claim: Up to unique isomorphism, there exists at most one topology on $C^\infty[a,b]$ such that to every compatible family of continuous linear maps $Z \to C^k[a,b]$ from a topological vector space Z there is a unique continuous linear $Z \to C^\infty[a,b]$ fitting into a commutative diagram as just above.

Proof: Let X, Y be $C^\infty[a,b]$ with two topologies fitting into such diagrams, and show $X \approx Y$, and for a unique isomorphism. First, claim that the identity map $\mathrm{id}_X : X \to X$ is the only map $\varphi : X \to X$ fitting into a commutative diagram

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[1] It is not essential to prove that there is no reasonable Banach space structure on $C^\infty[a,b]$, but this can be readily proven in a suitable context.

$$
\begin{tikzcd}
X \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
X \arrow[u, "\varphi"] \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^1[a,b] \arrow[r] \arrow[u, equal] & C^o[a,b] \arrow[u, equal]
\end{tikzcd}
$$

Indeed, given a compatible family of maps $X \to C^k[a,b]$, there is *unique* φ fitting into

$$
\begin{tikzcd}
X \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
& X \arrow[ul, dotted, "\varphi"] \arrow[ur, dashed] \arrow[urr, dashed, "\forall"'] & &
\end{tikzcd}
$$

Since the identity map id_X fits, necessarily $\varphi = \mathrm{id}_X$. Similarly, given the compatible family of inclusions $Y \to C^k[a,b]$, there is unique $f : Y \to X$ fitting into

$$
\begin{tikzcd}
X \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
& Y \arrow[ul, dotted, "f"] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

Similarly, given the compatible family of inclusions $X \to C^k[a,b]$, there is unique $g : X \to Y$ fitting into

$$
\begin{tikzcd}
Y \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
& X \arrow[ul, dotted, "g"] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

Then $f \circ g : X \to X$ fits into a diagram

$$
\begin{tikzcd}
X \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
& X \arrow[ul, dotted, "f\circ g"] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

Therefore, $f \circ g = \mathrm{id}_X$. Similarly, $g \circ f = \mathrm{id}_Y$. That is, f, g are mutual inverses, so are isomorphisms of topological vector spaces. ///

Existence of a topology on $C^\infty[a,b]$ satisfying the condition above will be proven by identifying $C^\infty[a,b]$ as the obvious diagonal *closed subspace* of the *topological product* of the *limitands* $C^k[a,b]$:

$$C^\infty[a,b] \;=\; \{\{f_k : f_k \in C^k[a,b]\} : f_k = f_{k+1} \text{ for all } k\}$$

An arbitrary *product* of topological spaces X_α for α in an index set A is a topological space X with (*projections*) $p_\alpha : X \to X_\alpha$, such that every family $f_\alpha : Z \to X_\alpha$ of maps from any other topological space Z *factors through* the p_α *uniquely*, in the sense that there is a unique $f : Z \to X$ such that $f_\alpha = p_\alpha \circ f$ for all α. Pictorially, *all triangles commute* in the diagram

$$
\begin{tikzcd}
Z \arrow[rrr, dotted, "f"] \arrow[dr, dashed, "f_\alpha"'] \arrow[drrrr, dashed, "f_\beta"] & & & X \arrow[dll, "p_\alpha"'] \arrow[dr, "p_\beta"] & & \\
\dots & X_\alpha & \dots & & X_\beta & \dots
\end{tikzcd}
$$

A similar argument to that for uniqueness of limits proves *uniqueness* of products up to unique isomorphism. *Construction* of products is by putting the usual product topology with basis consisting of products $\prod_\alpha Y_\alpha$ with $Y_\alpha = X_\alpha$ for all but finitely-many indices, on the Cartesian product of the *sets* X_α, whose existence we grant ourselves. Proof that this usual is *a* product amounts to unwinding the definitions. By uniqueness, in particular, despite the plausibility of the *box topology* on the product, it cannot function as a product topology since it differs from the standard product topology in general.

[2.0.2] Claim: Giving the diagonal copy of $C^\infty[a,b]$ inside $\prod_k C^k[a,b]$ the subspace topology yields a (projective) limit topology.

Proof: The projection maps $p_k : \prod_j C^j[a,b] \to C^k[a,b]$ from the whole product to the factors $C^k[a,b]$ are continuous, so their restrictions to the diagonally imbedded $C^\infty[a,b]$ are continuous. Further, letting $i_k : C^k[a,b] \to C^{k-1}[a,b]$ be the inclusion, on that diagonal copy of $C^\infty[a,b]$ we have $i_k \circ p_k = p_{k-1}$ as required.

On the other hand, *any* family of maps $\varphi_k : Z \to C^k[a,b]$ induces a map $\widetilde{\varphi} : Z \to \prod C^k[a,b]$ such that $p_k \circ \widetilde{\varphi} = \varphi_k$, by the property of the product. *Compatibility* $i_k \circ \varphi_k = \varphi_{k-1}$ implies that the image of $\widetilde{\varphi}$ is inside the diagonal, that is, inside the copy of $C^\infty[a,b]$. ///

A *countable* product of *metric* spaces X_k with metrics d_k has no canonical single metric, but is *metrizable*. One of many topologically equivalent metrics is the usual

$$d(\{x_k\},\{y_k\}) \;=\; \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k - y_k)}{d_k(x_k - y_k) + 1}$$

When the metric spaces X_k are *complete*, the product is complete. A closed subspace of a complete metrizable space is complete metrizable, so we have

[2.0.3] Corollary: $C^\infty[a,b]$ is complete metrizable. ///

Abstracting the above, for a (not necessarily countable) family

$$\ldots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_o$$

of Banach spaces with continuous linear transition maps as indicated, *not* recessarily requiring the continuous linear maps to be injective (or surjective), a *(projective) limit* $\lim_i B_i$ is a topological vector space with continuous linear maps $\lim_i B_i \to B_j$ such that, for every compatible family of continuous linear maps $Z \to B_i$ there is unique continuous linear $Z \to \lim_i B_i$ fitting into

$$\begin{tikzcd}
\lim_i B_i \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots \arrow[r, "\varphi_2"] & B_1 \arrow[r, "\varphi_1"] & B_o \\
 & Z \arrow[ul, dotted] \arrow[ur, dashed] \arrow[urr, dashed] & &
\end{tikzcd}$$

The same *uniqueness* proof as above shows that there is at most one topological vector space $\lim_i B_i$. For *existence* by *construction*, the earlier argument needs only minor adjustment. The conclusion of complete metrizability would hold when the family is countable.

Before declaring $C^\infty[a,b]$ to be a *Fréchet* space, we must certify that it is *locally convex*, in the sense that every point has a local basis of *convex* opens. Normed spaces are immediately locally convex, because open balls are convex: for $0 \le t \le 1$ and x, y in the ε-ball at 0 in a normed space,

$$|tx + (1-t)y| \;\le\; |tx| + |(1-t)y| \;\le\; t|x| + (1-t)|y| \;<\; t\cdot\varepsilon + (1-t)\cdot\varepsilon \;=\; \varepsilon$$

Product topologies of locally convex vectorspaces are locally convex, from the *construction* of the product. The construction of the limit as the diagonal in the product, with the subspace topology, shows that it is locally convex. In particular, *countable limits of Banach spaces are locally convex, hence, are Fréchet.* All spaces of practical interest are locally convex for simple reasons, so demonstrating local convexity is rarely interesting.

[2.0.4] Theorem: $\frac{d}{dx} : C^\infty[a,b] \to C^\infty[a,b]$ is continuous.

Proof: In fact, the differentiation operator is characterized via the expression of $C^\infty[a,b]$ as a limit. We already know that differentiation d/dx gives a continuous map $C^k[a,b] \to C^{k-1}[a,b]$. Differentiation is compatible with the inclusions among the $C^k[a,b]$. Thus, we have a commutative diagram

$$\begin{tikzcd}
C^\infty[a,b] \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots & C^k[a,b] \arrow[r] & C^{k-1}[a,b] \arrow[r] & \ldots \\
C^\infty[a,b] \arrow[rr, bend right] \arrow[rrr, bend right] & \ldots & C^k[a,b] \arrow[r] \arrow[ur, "\frac{d}{dx}"] & C^{k-1}[a,b] \arrow[r] \arrow[ur, "\frac{d}{dx}"] & \ldots
\end{tikzcd}$$

Composing the projections with d/dx gives (dashed) induced maps from $C^\infty[a,b]$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
C^\infty[a,b] \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots & C^k[a,b] \arrow[r] & C^{k-1}[a,b] \arrow[r] & \ldots \\
C^\infty[a,b] \arrow[u, dotted, "\frac{d}{dx}"] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[urrrr, dashed] \arrow[rr, bend right] \arrow[rrr, bend right] & \ldots & C^k[a,b] \arrow[r] \arrow[ur] & C^{k-1}[a,b] \arrow[r] \arrow[ur] & \ldots
\end{tikzcd}$$

This proves the continuity of differentiation in the limit topology. ///

In a slightly different vein, we have

[2.0.5] Claim: For all $x \in [a,b]$ and for all non-negative integers k, the evaluation map $f \to f^{(k)}(x)$ is a continuous linear map $C^\infty[a,b] \to \mathbb{C}$.

Proof: The inclusion $C^\infty[a,b] \to C^k[a,b]$ is continuous, and the evaluation of the k^{th} derivative is continuous. ///