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07. *Distributions = generalized functions*

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Completeness is a very desirable property for a topology on a space of functions, since it means exactly that we can take limits in that topology without falling outside that class of functions.

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1. *Fréchet space* $C^\infty[a,b]$

The space $C^\infty[a,b]$ of infinitely differentiable complex-valued functions on a (finite) interval $[a,b]$ in $\mathbb{R}$ is *not* a Banach space. [1] Nevertheless, the topology is *completely determined* by its relation to the Banach spaces $C^k[a,b]$, and is *complete metric*, although the metric is *not canonical*. Since

$$C^\infty[a,b] \;=\; \bigcap_{k\ge 0} C^k[a,b]$$

we reasonably require (at least) that every inclusion $C^\infty[a,b] \longrightarrow C^k[a,b]$ is continuous. A natural way to do this is to give $C^\infty[a,b]$ a *family* of (semi-)[2] norms, each of which gives a *pseudo-*[3] metric, which *altogether* give a metric (which we subsequently show is complete). Namely, let

$$\nu_k(f) \;=\; \sup_{x\in[a,b]} |f^{(k)}(x)| \qquad\qquad (\text{for } f\in C^\infty[a,b],\text{ for } 0\le k\in\mathbb{Z})$$

Not all these are actual *norms*: for example, for non-zero constant f, f' is identically 0 and $\nu_1(f) = 0$ although f is not the 0 function. Still, these seminorms do retain the desirable *homogeneity* property $\nu_k(t\cdot f| = |t|_{\mathbb{C}}\cdot \nu_k(f)$ for constants g. This *family* of seminorms ν_k is *separating* since $\nu_k(f) = 0$ for all k implies $f = 0$. (In fact, $\nu_0(f) = 0$ already implies $f = 0$, but that's an irrelevant piece of luck.)

The topology specified by a (separating) family of seminorms $\{\nu_k\}$ has *local sub-basis*[4] at 0 consisting of sets

$$U_{k,\varepsilon} \;=\; \{f : \nu_k(f) < \varepsilon\} \qquad\qquad (\text{for all indices } k \text{ and all } \varepsilon > 0)$$

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[1] It is not essential to prove that there is no reasonable Banach space structure on $C^\infty[a,b]$, but this can be readily proven in a suitable context.

[2] A *semi-norm* $\nu()$ has the properties of a (genuine) norm, except that $\nu(f) = 0$ need not imply $f = 0$.

[3] A *pseudo-metric* $p(,)$ has the properties of a metric, except that $p(x,y) = 0$ need not imply that $x = y$.

[4] Recall that a local sub-basis at a point x_o in a topological space is a set of opens whose *finite intersections* are a *local basis* at x_o.

and local sub-basis at g consisting of sets $g + U_{k,\varepsilon}$. This does not use the countability of the index set. Every such topology is *locally convex*, in the sense that every point has a local basis consisting of *convex* opens. [5] Normed spaces are immediately locally convex, because open balls are convex, and the argument applies to the sub-basis opens specified by seminorms ν: for $0 \le t \le 1$ and x, y in $U_\varepsilon = \{z : \nu(z) < \varepsilon\}$,

$$\nu(tx + (1-t)y) \;\le\; \nu(tx) + \nu(1-t)y) \;\le\; |t|\nu(x) + |1-t|\nu(y) \;<\; t \cdot \varepsilon + (1-t) \cdot \varepsilon \;=\; \varepsilon$$

Intersections of convex sets are convex, so there is a local basis consisting of opens. [6]

For each seminorm ν_k make a corresponding *pseudo-metric* $p_k(f,g) = \nu_k(f-g)$. Recall that a *countable* product of *metric* spaces X_k with metrics d_k has no canonical single metric, but is *metrizable*, with a choice from many topologically equivalent metrics the usual

$$d(\{x_k\},\{y_k\}) \;=\; \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k,y_k)}{d_k(x_k,y_k)+1}$$

Further, when the metric spaces X_k are *complete*, the product with this metric is complete. Similarly, give $C^\infty[a,b]$ the (genuine) metric made from the pseudo-metrics p_k:

$$d(f,g) \;=\; \sum_{k=0}^{\infty} 2^{-k} \frac{p_k(f,g)}{p_k(f,g)+1}$$

[1.1] Claim: With this metric $C^\infty[a,b]$ is complete metrizable.

Proof: *(Sketch)* Imbed $C^\infty[a,b]$ on the diagonal in the countably infinite $\prod_k C^k[a,b]$, and show that it is a *closed* subset in the product topology. A closed subspace of a complete metrizable space is complete metrizable. ///

A vector space with a topology is a *Fréchet* space when it is complete metric with a *translation-invariant* metric, meaning that $d(f+h, g+h) = d(f,g)$ as is the case here, and is locally convex. Again, local convexity is guaranteed for topologies described by (separating) families of seminorms, countable or not. Thus, $C^\infty[a,b]$ is a Fréchet space.

By design,

[1.2] Claim: $\frac{d}{dx} : C^\infty[a,b] \to C^\infty[a,b]$ is continuous.

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[5] Although we only care about locally convex topological vectorspaces, there *do exist* complete-metric topological vectorspaces which *fail* to be locally convex. This underscores the need to specify that a Fréchet space should be locally convex. The usual example of a not-locally-convex complete-metric space is the sequence space

$$\ell^p \;=\; \{x = (x_1, x_2, \ldots) \;:\; \sum_i |x_i|^p < \infty\}$$

for $0 < p < 1$ with metric

$$d(x,y) \;=\; \sum_i |x_i - y_i|^p \qquad\qquad \text{(note: no } p^{th} \text{ root, unlike the } p \ge 1 \text{ case)}$$

This example's interest is mostly as a counterexample to a naive presumption that local convexity is automatic.

[6] All spaces of practical interest are locally convex for simple reasons, so demonstrating local convexity is rarely interesting.

Proof: [... iou ...] ///

2. *Fréchet space of smooth functions* $\mathcal{E} = C^\infty(\mathbb{R}^n)$

For a *non-compact* topological space such as $\mathbb{R}^n$, the space $C^o(\mathbb{R}^n)$ of continuous functions is *not* a Banach space with sup norm, because the sup of the absolute value of a continuous function may be $+\infty$. Nevertheless, sups of continuous functions on *compacts* are finite, which suggests a collection of *seminorms* giving a good topology, namely,

$$\nu_K(f) \;=\; \sup_{x\in K} |f(x)| \qquad\qquad (\text{for compact } K \subset \mathbb{R}^n)$$

Further, since $\mathbb{R}^n$ can easily be expressed as a *countable* union of compact subsets, for example, closed balls of integer radius, there is a *countable* (still separating) sub-collection of seminorms giving the same topology. Thus, this topology is given (non-canonically) by a *metric*. Since the topology is given by seminorms, it is locally convex.

[2.1] Claim: $C^o(\mathbb{R}^n)$ is *complete*, so is a Fréchet space.

Proof: A sequence $\{f_n\}$ is *Cauchy* if and only if $\{f_n|_K\}$ is Cauchy for every compact K. We already know that $C^o(K)$ is complete, so the uniform pointwise limit of $\{f_n|_K\}$ is continuous on K. The pointwise limits for various K are compatible, so the uniform-on-compacts pointwise limit of f_n is in $C^o(\mathbb{R}^n)$. ///

A similar device applies to $\mathcal{E} = C^\infty(\mathbb{R}^n)$. For compact K and index k, let

$$\nu_{K,k}(f) \;=\; \sup_{x\in K}\sup_{|\alpha|\le k} |f^{(\alpha)}(x)| \qquad\qquad (\text{for } f \in C^\infty(\mathbb{R}^n))$$

where α runs over multi-indices $\alpha = (\alpha_1,\ldots,\alpha_n)$, $|\alpha| = \sum_i \alpha_i$, and in the usual conventions partial derivatives are denoted

$$f^{(\alpha)} \;=\; \frac{\partial^\alpha f}{\partial x^\alpha} \;=\; \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$$

These (separating) seminorms give a locally convex topology. Since $\mathbb{R}^n$ is a countable union of compacts, there is a countable collection of seminorms giving the same topology, which is therefore metrizable.

[2.2] Theorem: $\mathcal{E}$ is complete, so is Fréchet.

Proof: [... iou ...] ///

[2.3] Claim: $\frac{\partial}{\partial x_i} : \mathcal{E} \to \mathcal{E}$ is continuous.

Proof: [... iou ...] ///

3. *Fréchet space of Schwartz functions* $\mathscr{S}$ *on* $\mathbb{R}^n$

The space of Schwartz functions turns out to be an astute choice of minimal collection of functions stable under Fourier transform, hence its importance.

A continuous function f on $\mathbb{R}^n$ is *of rapid decay* when

$$\sup_{x\in\mathbb{R}} (1+|x|^2)^n \cdot |f(x)| \;<\; +\infty \qquad\qquad (\text{for every } n = 1, 2, \ldots)$$

The space of *Schwartz functions* is

$$\mathscr{S}(\mathbb{R}) \;=\; \{\text{smooth functions } f \text{ all whose derivatives are of rapid decay}\}$$

A reasonable topology on $\mathscr{S}(\mathbb{R})$ is given by seminorms

$$\nu_{k,N}(f) \;=\; \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\le k} (1+|x|^2)|f^\alpha(x)|$$

where α runs over multi-indices. Being given by seminorms, the topology is locally convex. Since the family of seminorms is *countable*, the topology is metrizable.

[3.1] Claim: $\mathscr{S}(\mathbb{R})$ is complete, hence Fréchet.

Proof: [... iou ...] ///

We have already shown

[3.2] Claim: Fourier transform is a continuous map $\mathscr{S} \to \mathscr{S}$. ///

[3.3] Claim: $\frac{\partial}{\partial x_j} : \mathscr{S} \to \mathscr{S}$ is *continuous.*

Proof: It suffices, given N, k, and $\varepsilon > 0$, to find $N', k', \delta > 0$ such that

$$\nu_{N',k'}(f) < \delta \implies \nu_{N,k}(\frac{\partial}{\partial x_i} f) < \varepsilon$$

To this end, it suffices to take $N' = N$, $k' = k+1$, and $\delta = \varepsilon$, since

$$\nu_{N,k}(\frac{\partial}{\partial x_i} f) \;\le\; \nu_{N,k+1}(f)$$

giving the desired continuity. ///

[3.4] Claim: Compactly-supported smooth functions are *dense* in $\mathscr{S}$.

Proof: [... iou ...] ///

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4. *LF-space of test functions $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$*

Here are examples of somewhat subtler types of topological vector spaces. Although they are provably locally convex, we do not give their topologies by seminorms.

First, consider a slightly simpler example, the space of compactly-supported *continuous* functions

$$C_c^o(\mathbb{R}^n) \;=\; \text{compactly-supported continuous functions on } \mathbb{R}^n$$

It is an *ascending union* of the subspaces

$$C_N^o \;=\; \{f \in C^o(\mathbb{R}^n) \;:\; \text{spt} f \subset \{x : |x| \le N\}$$

Each space C_N^o is a Banach space, being a closed subspace of the Banach space $C^o(N\text{-ball})$, so is a Banach space. However, each C_N^o is closed and *nowhere dense* in C_{N+1}^o, meaning that C_N^o has empty interior C_{N+1}^o. Thus, it appears that $C_c^o(\mathbb{R}^n)$ is probably a countable union of nowhere dense subsets, which is the negation of the conclusion of Baire's theorem, which suggests that a reasonable topology on $C_c^o(\mathbb{R}^n)$ *cannot be complete metric.* [7]

Yes, this is disquieting, but $C_c^o(\mathbb{R})$ is not a pathological space.

Being a countable ascending union of the Banach spaces C_N^o (which are also Fréchet spaces), it is called an *LF-space* for (*(strict inductive/co-) limit-of-Fréchet*). Such spaces are

Again, hopefully agreeing that $C_c^o(\mathbb{R})$ is not a pathological space, we note some apparently unavoidable complications. For example, while in a metric space every Cauchy *net* actually has a Cauchy sub-*sequence* converging to the same limit, which makes consideration of sequences adequate in that context, for a not-metric topology there is no such guarantee that information about Cauchy *nets* can be captured by *sequences.* In particular, it might happen that some Cauchy nets might fail to converge while all Cauchy sequences *do* converge. That is, the strongest or fullest requirement of *completeness* is an unreasonably strong requirement outside the case of metric spaces.

Nevertheless, a weaker notion of completeness, *quasi-* or *local* completeness, does hold in all reasonable examples of interest, *and* is sufficient for nearly all applications. We will return to a more careful discussion of this later. [8] Noting that we have not quite given any clear description of the LF-space topology on $C_c^o(\mathbb{R}^n)$, nevertheless we state

[4.1] Theorem: $C_c^o(\mathbb{R}^n)$ with its LF-space topology is *quasi-complete. [... iou ...]*

Similarly, the space of *test functions*

$$\mathcal{D} \;=\; C_c^\infty(\mathbb{R}^n) \;=\; \text{compactly-supported smooth functions on } \mathbb{R}^n$$

is an *ascending union* of the subspaces

$$C_N^\infty \;=\; \{f \in C^\infty(\mathbb{R}^n) \;:\; \text{spt} f \subset \{x : |x| \le N\}$$

Each space C_N^∞ is a Frechet space, being a closed subspace of the Frechet space $C^\infty(N\text{-ball})$. Again, each C_N^∞ is closed and *nowhere dense* in C_{N+1}^∞, so it appears that $C_c^\infty(\mathbb{R}^n)$ is probably a countable union of nowhere dense subsets, which is the negation of the conclusion of Baire's theorem, which suggests that a reasonable topology on $C_c^o(\mathbb{R}^n)$ *cannot be complete metric.*

[4.2] Theorem: $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$ is an LF-space, and is *quasi-complete. [... iou ...]*

[4.3] Claim: The inclusion $\mathcal{D} \to \mathscr{S}$ is *continuous*, and the image is *dense. [... iou ...]*

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5. *Generalized functions (distributions) $\mathcal{D}^*$, $\mathscr{S}^*$, $\mathcal{E}^*$ as linear functionals*

The most immediate definition of the space $\mathcal{D}^*$ of *distributions* or *generalized functions* on $\mathbb{R}^n$ is as the dual $\mathcal{D}^* = \mathcal{D}(\mathbb{R})^* = C_c^\infty(\mathbb{R})^*$ to the space $\mathcal{D}$ of test functions, with the *weak dual topology* (see appendix) given by seminorms $\nu_f(u) = |u(f)|$ for test functions f and distributions u.

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[7] If we give $C_c^o(\mathbb{R}^n)$ the sup norm, it is not complete. Its completion is $C_o^o(\mathbb{R}^n)$, the Banach space of continuous functions going to 0 at infinity. This is an eminently reasonable space, but if we want $C_c^o(\mathbb{R}^n)$ to already be complete, it is not. Similarly, we could give $C_c^o(\mathbb{R}^n)$ the L^2 norm, but it would not be complete, and its completion would be L^2. Yes, locally countably-based spaces are *metrizable*, but not generally *complete* with that metric.

[8] The simplest example of definitive failure of full completeness is that the *weak dual* of ℓ^2 is *not* complete: there are Cauchy nets that do not converge. Nevertheless, that weak dual is *quasi-* complete.

Similarly, the space of *tempered* distributions is $\mathscr{S}^*$ with weak dual topology. The space of *compactly-supported* distributions $\mathcal{E}^*$ is $\mathcal{E}^*$ with the weak dual topology. Naming $\mathcal{E}^*$ *compactly-supported* will be justified later.

Because taking (continuous) duals is inclusion-reversing, the continuous containments $\mathcal{D} \subset \mathscr{S} \subset \mathcal{E}$ give continuous maps $\mathcal{E}^* \to \mathscr{S}^* \to \mathcal{D}^*$.

When we know that $\mathcal{D}$ is *dense* in $\mathscr{S}$ and in $\mathcal{E}$, it will follow that these are *injections*. Without that density and injectivity, we could not claim that $\mathcal{E}^*$ and $\mathscr{S}^*$ *are* distributions, but only that they naturally *map to* distributions.

Despite not being complete in the strongest possible sense, $\mathcal{D}^*$, $\mathscr{S}^*$, and $\mathcal{E}^*$ are provably *quasi-complete*, which suffices for applications.

The description of spaces of distributions as the (weak) duals to various spaces of functions falls far short of explaining its utility. There is a natural imbedding $\mathcal{D} \to \mathcal{D}^*$ of test functions into distributions, by

$$f \to u_f \qquad \text{by} \qquad u_f(g) = \int_{\mathbb{R}^n} f(x)\, g(x)\, dx \qquad (\text{for } f, g \in \mathcal{D})$$

That is, via this imbedding we consider distributions to be *generalized functions*. Indeed, later we will see that test functions $\mathcal{D}$ are *dense* in $\mathcal{D}^*$ (of course in the topology of the latter).

The simplest example of a distribution not obtained by integration against a test function on $\mathbb{R}$ is the *Dirac delta*, the evaluation map $\delta(f) = f(0)$, which is *continuous* for the LF-space topology on test functions.

The imbedding $\mathcal{D} \to \mathcal{D}^*$ and insistence on compatilibty with integration by parts, explain how to define $\frac{\partial}{\partial x_i}$ on distributions in a form compatible with the imbedding $\mathcal{D} \subset \mathcal{D}^*$: noting the sign, due to integration by parts,

$$\Big(\frac{\partial}{\partial x_i} u\Big)(f) \;=\; -\,u\Big(\frac{\partial}{\partial x_i} f\Big) \qquad (\text{for } u \in \mathcal{D}^* \text{ and } f \in \mathcal{D})$$

[5.1] Claim: $\frac{d}{dx} : \mathcal{D}^* \to \mathcal{D}^*$ is continuous.

Proof: By the nature of the weak dual topology, it suffices to show that for each $f \in \mathcal{D}$ and $\varepsilon > 0$ there are $g \in \mathcal{D}$ and $\delta > 0$ such that $|u(g)| < \delta$ implies $|(\frac{\partial}{\partial x_i} u)(f)| < \varepsilon$. Taking $g = \frac{\partial}{\partial x_i} f$ and $\delta = \varepsilon$ succeeds. ///

The *order* of a distribution $u : \mathcal{D} \to \mathbb{C}$ is the integer k, if such exists, such that u is continuous when $\mathcal{D}$ is given the weaker topology from $\text{colim}_K C^k_K$. Not every distribution has finite order, but there is a useful technical application of the previous discussion:

[5.2] Corollary: A distribution $u \in \mathcal{D}^*$ with *compact support* has *finite order*.

Proof: Let ψ be a test function that is identically 1 on an open containing the support of u. Then

$$u(f) \;=\; u((1-\psi)\cdot f) + u(\psi \cdot f) \;=\; 0 + u(\psi \cdot f)$$

since $(1-\psi)\cdot f$ is a test function with support not meeting the support of u. With $K = \text{spt}\,\psi$, this suggests that u factors through a subspace of $\mathcal{D}_K$ via $f \to \psi \cdot f \to u(\psi \cdot f)$, but there is the issue of continuity.

///

[5.3] Claim: In the inclusion $\mathcal{E}^* \subset \mathscr{S}^* \subset \mathcal{D}^*$, the image of $\mathcal{E}^*$ really is the collection of distributions with compact support.

Proof: *[... iou ...]* ///

6. Generalized functions $\mathcal{D}^*$, $\mathscr{S}^*$, $\mathcal{E}^*$ as completions of $\mathcal{D}$

[iou]

7. Fourier transforms of tempered distributions

Since $\mathscr{S}$ is mapped to itself by Fourier transform [13.13], this gives a way to define Fourier transform on $\mathscr{S}^*$, by a duality extending the Plancherel theorem:

$$\widehat{u}(f) \;=\; u(\widehat{f}) \qquad\qquad (\text{for } f \in \mathscr{S} \text{ and } u \in Schw^*)$$

[7.1] Corollary: Fourier transform is a homeomorphism of the space tempered distributions $\mathscr{S}^*$ to itself, by

$$\widehat{u}(\varphi) \;=\; u(\widehat{\varphi}) \qquad\qquad (\text{for all } \varphi \in \mathscr{S})$$

Proof: Fourier transform is a topological isomorphism of $\mathscr{S}$ to itself. ///

8. Distributions supported at a point

Recall that the support of a *function* is the *closure* of the set on which it is non-zero, slightly complicating the notion of support for a *distribution* u: support of u is the *complement* of the *union* of all open sets U such that $u(f) = 0$ for all test functions f with support inside U.

[8.1] Theorem: A distribution with *support* $\{0\}$ is a finite linear combination of Dirac's δ and its derivatives.

Proof: Since $\mathcal{D}$ is a colimit of $\mathcal{D}_K$ over $K = [-n, n]$, it suffices to classify u in $\mathcal{D}_K^*$ with support $\{0\}$. We claim that a continuous linear functional on $\mathcal{D}_K = \lim_k C_K^k$ factors through some limitand

$$C_K^k \;=\; \{f \in C^k(K) :\; f^{(i)} \text{ vanishes on } \partial K \text{ for } 0 \le i \le k\}$$

This is a special case of

[8.2] Claim: Let $X = \lim_n B_n$ be a limit of Banach spaces, with the image of X *dense* in each B_n. A continuous linear map $T : \lim_n B_n \to Z$ to a *normed* space Z factors through some limitand B_n. For $Z = \mathbb{C}$, the same conclusion holds without the density assumption.

Proof: Let $X = \lim_i B_i$ with projections $p_i : X \to B_i$. Each B_i is the closure of the image of X. By the continuity of T at 0, there is an open neighborhood U of 0 in X such that TU is inside the open unit ball at 0 in Z. By the description of the limit topology as the product topology restricted to the diagonal, there are finitely-many indices $i_1, \ldots, i_n$ and open neighborhoods V_{i_t} of 0 in B_{i_t} such that

$$\bigcap_{t=1}^{n} p_{i_t}^{-1}\big(p_{i_t} X \cap V_{i_t}\big) \;\subset\; U$$

We can make a *smaller* open in X by a condition involving a single limitand, as follows. Let j be *any* index with $j \ge i_t$ for all t, and

$$N \;=\; \bigcap_{t=1}^{n} p_{i_t,j}^{-1}\big(p_{i_t,j} B_j \cap V_{i_t}\big) \;\subset\; B_j$$

By the compatibility $p_{i_t}^{-1} = p_j^{-1} \circ p_{i_t,j}^{-1}$, we have $p_{i_t,j} N \subset V_{i_t}$ for $i_1, \ldots, i_n$, and $p_j^{-1}(p_j X \cap N) \subset U$. By the linearity of T, for any $\varepsilon > 0$,

$$T(\varepsilon \cdot p_j^{-1}\big(p_j X \cap N)\big) \;=\; \varepsilon \cdot T(p_j^{-1}\big(p_j X \cap N)\big) \;\subset\; \varepsilon\text{-ball in } Z$$

We claim that T factors through $p_j X$ with the subspace topology from B_j. One potential issue in general is that $p_j : X \to B_j$ can have a non-trivial kernel, and we must check that $\ker p_j \subset \ker T$. By the linearity of T,

$$T(\frac{1}{n} \cdot p_j^{-1}(p_j \cap N)) \;\subset\; \frac{1}{n}\text{-ball in } Z$$

so

$$T\left(\bigcap_n \frac{1}{n} \cdot p_j^{-1}(p_j X \cap N)\right) \;\subset\; \frac{1}{m}\text{-ball in } Z \qquad\qquad \text{(for all } m\text{)}$$

and then

$$T\left(\bigcap_n \frac{1}{n} \cdot p_j^{-1}(p_j \cap N)\right) \;\subset\; \bigcap_m \frac{1}{m}\text{-ball in } Z \;=\; \{0\}$$

Thus,

$$\bigcap_n p_j^{-1}(p_j X \cap \frac{1}{n} \cdot N) \;=\; \bigcap_n \frac{1}{n} \cdot p^{-1}(p_j X \cap N) \;\subset\; \ker T$$

Thus, for $x \in X$ with $p_j x = 0$, certainly $p_j x \in \frac{1}{n} N$ for all $n = 1, 2, \ldots$, and

$$x \;\in\; \bigcap_n p_j^{-1}(p_j X \cap \frac{1}{n} N) \;\subset\; \ker T$$

This proves the subordinate claim that T factors through $p_j : X \to B_j$ via a (not necessarily continuous) linear map $T' : p_j X \to Z$. The continuity follows from continuity at 0, which is

$$T(\varepsilon \cdot p_j^{-1}\big(p_j X \cap N)\big) \;=\; \varepsilon \cdot T(p_j^{-1}\big(p_j X \cap N)\big) \;\subset\; \varepsilon\text{-ball in } Z$$

Then $T' : p_j X \to Z$ extends to a map $B_j \to Z$ by continuity: given $\varepsilon > 0$, take symmetric convex neighborhood U of 0 in B_j such that $|T'y|_Z < \varepsilon$ for $y \in p_j X \cap U$. Let y_i be a Cauchy net in $p_j X$ approaching $b \in B_j$. For y_i and y_j inside $b + \frac{1}{2}U$, $|T'y_i - T'y_j| = |T'(y_i - y_j)| < \varepsilon$, since $y_i - y_j \in \frac{1}{2} \cdot 2U = U$. Then unambiguously define $T'b$ to be the Z-limit of the $T'y_i$. The closure of $p_j X$ in B_j is B_j, giving the desired map.

When u is a *functional*, that is, a map to $\mathbb{C}$, we can extend it by Hahn-Banach. ///

Returning to the proof of the theorem: thus, there is $k \geq 0$ such that u factors through a limitand C_K^k. In particular, u is continuous for the C^k topology on $\mathcal{D}_K$.

We need an auxiliary gadget. Fix a test function ψ identically 1 on a neighborhood of 0, bounded between 0 and 1, and (necessarily) identically 0 outside some (larger) neighborhood of 0. For $\varepsilon > 0$ let

$$\psi_\varepsilon(x) \;=\; \psi(\varepsilon^{-1} x)$$

Since the support of u is just $\{0\}$, for all $\varepsilon > 0$ and for all $f \in \mathcal{D}(\mathbb{R}^n)$ the support of $f - \psi_\varepsilon \cdot f$ does not include 0, so

$$u(\psi_\varepsilon \cdot f) \;=\; u(f)$$

Thus, for implied constant depending on k and K, but not on f,

$$|\psi_\varepsilon f|_k \;=\; \sup_{x \in K} \sum_{0 \leq i \leq k} |(\psi_\varepsilon f)^{(i)}(x)| \;\ll\; \sum_{i \leq k} \sum_{0 \leq j \leq i} \sup_x \varepsilon^{-j} \left|\psi^{(j)}(\varepsilon^{-1} x)\, f^{(i-j)}(x)\right|$$

For test function f vanishing to order k at 0, that is, $f^{(i)}(0) = 0$ for all $0 \le i \le k$, on a fixed neighborhood of 0, by a Taylor-Maclaurin expansion, $|f(x)| \ll |x|^{k+1}$, and, generally, for i^{th} derivatives with $0 \le i \le k$, $|f^{(i)}(x)| \ll |x|^{k+1-i}$. By design, all derivatives $\psi', \psi'', \ldots$ are identically 0 in a neighborhood of 0, so, for suitable implied constants independent of ε,

$$|\psi_\varepsilon f|_k \ll \sum_{0\le i\le k} \sum_{0\le j\le i} \varepsilon^{-j} \cdot \left|\psi^{(j)}(\varepsilon^{-1}x)\, f^{(i-j)}(x)\right| \ll \sum_{0\le i\le k} \sum_{j=0} \varepsilon^{-j} \cdot 1 \cdot \varepsilon^{k+1-i}$$

$$= \sum_{0\le i\le k} \varepsilon^{k+1-i} \ll \varepsilon^{k+1-k} = \varepsilon$$

Thus, for sufficiently small $\varepsilon > 0$, for smooth f vanishing to order k at 0, $|u(f)| = |u(\psi_\varepsilon f)| \ll \varepsilon$, and $u(f) = 0$. That is,

$$\ker u \supset \bigcap_{0\le i\le k} \ker \delta^{(i)}$$

The conclusion, that u is a linear combination of the distributions $\delta, \delta', \delta^{(2)}, \ldots, \delta^{(k)}$, follows from

[8.3] Claim: A linear functional $\lambda \in V^*$ vanishing on the intersection $\bigcap_i \ker \lambda_i$ of kernels of a finite collection $\lambda_1, \ldots, \lambda_n \in V^*$ is a *linear combination* of the λ_i.

Proof: The linear map

$$q : V \longrightarrow \mathbb{C}^n \quad \text{by} \quad v \longrightarrow (\lambda_1 v, \ldots, \lambda_n v)$$

is *continuous* since each λ_i is continuous, and λ factors through q, as $\lambda = L \circ q$ for some linear functional L on $\mathbb{C}^n$. We know all the linear functionals on $\mathbb{C}^n$, namely, L is of the form

$$L(z_1, \ldots, z_n) = c_1 z_1 + \ldots + c_n z_n \qquad \text{(for some } c_i \in \mathbb{C}\text{)}$$

Thus,

$$\lambda(v) = (L \circ q)(v) = L(\lambda_1 v, \ldots, \lambda_n v) = c_1 \lambda_1(v) + \ldots + c_n \lambda_n(v)$$

expressing λ as a linear combination of the λ_i. ///

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9. *Appendix: weak dual topologies*

For a topological vectorspace V, with (continuous) *dual*

$$V^* = \{\text{continuous linear maps } V \to \mathbb{C}\}$$

the *weak dual topology*[9] on V^* has a local sub-basis at 0 consisting of sets

$$U = U_{v,\varepsilon} = \{\lambda \in V^* : |\lambda(v)| < \varepsilon\} \qquad \text{(for fixed } v \in V \text{ and } \varepsilon > 0)$$

That is, the topology is given by seminorms $\nu_v(\lambda) = |\lambda(v)|$. Unless V is finite-dimensional, this topology on V^* is much coarser than a Banach, Fréchet, or LF-topology. The map $\lambda \to |\lambda(v)|$ is a natural example of a *seminorm*: it is not a norm, because $\lambda(v) = 0$ can easily happen without $v = 0$.

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[9] The weak dual topology is traditionally called the *weak-∗-topology*, but replacing ∗ by *dual* is more explanatory.