(*February 24, 2017*)

08b. Examples of spectra of operators

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/real/notes_2016-17/08b-examples_of_spectra.pdf]

The usefulness of the notion of spectrum of an operator on a Hilbert space is the analogy to eigenvalues of operators on finite-dimensional spaces. Naturally, things become more complicated in infinite-dimensional vector spaces.

---

1. *Generalities on spectra*

It is convenient to know that spectra of continuous operators are *non-empty*, *compact* subsets of $\mathbb{C}$.

Knowing this, *every* non-empty compact subset of $\mathbb{C}$ is easily made to appear as the spectrum of a continuous operator, even *normal* ones, as below.

[1.1] Proposition: The spectrum $\sigma(T)$ of a continuous linear operator $T : V \to V$ on a Hilbert space V is *bounded* by the operator norm $|T|_{\mathrm{op}}$.

Proof: For $|\lambda| > |T|_{\mathrm{op}}$, an obvious heuristic suggests an expression for the *resolvent* $R_\lambda = (T-\lambda)^{-1}$:

$$(T-\lambda)^{-1} \;=\; -\lambda^{-1}\cdot\big(1-\frac{T}{\lambda}\big)^{-1} \;=\; -\lambda^{-1}\cdot\Big(1+\frac{T}{\lambda}+\big(\frac{T}{\lambda}\big)^2+\ldots\Big)$$

The infinite series converges in operator norm for $|T/\lambda|_{\mathrm{op}} < 1$, that is, for $|\lambda| > |T|_{\mathrm{op}}$. Then

$$(T-\lambda)\cdot(-\lambda^{-1})\cdot\Big(1+\frac{T}{\lambda}+\big(\frac{T}{\lambda}\big)^2+\ldots\Big) \;=\; 1$$

giving a continuous inverse $(T-\lambda)^{-1}$, so $\lambda \notin \sigma(T)$. ///

[1.2] Remark: The same argument applied to T^n shows that $\sigma(T^n)$ is inside the closed ball of radius $|T^n|_{\mathrm{op}}$. By the elementary identity

$$T^n - \lambda^n \;=\; (T-\lambda)\cdot(T^{n-1}+T^{n-2}\lambda+\ldots+T\lambda^{n-2}+\lambda^{n-2})$$

$(T-\lambda)^{-1}$ exists for $|\lambda^n| > |T^n|_{\mathrm{op}}$, that is, for $|\lambda| > |T^n|_{\mathrm{op}}^{1/n}$. That is, $\sigma(T)$ is inside the closed ball of radius $\inf_{n\ge 1}|T^n|_{\mathrm{op}}^{1/n}$. The latter expression is the *spectral radius* of T. This notion is relevant to *non-normal* operators, such as the *Volterra operator*, whose spectral radius is 0, while its operator norm is much larger.

[1.3] Proposition: The spectrum $\sigma(T)$ of a continuous linear operator $T : V \to V$ on a Hilbert space V is *compact*.

Proof: That $\lambda \notin \sigma(T)$ is that there is a continuous linear operator $(T-\lambda)^{-1}$. We claim that for μ sufficiently close to λ, $(T-\mu)^{-1}$ exists. Indeed, a heuristic suggests an expression for $(T-\mu)^{-1}$ in terms of $(T-\lambda)^{-1}$. The algebra is helpfully simplified by replacing T by $T+\lambda$, so that $\lambda = 0$. With μ near 0 and granting existence of T^{-1}, the heuristic is

$$(T-\mu)^{-1} \;=\; (1-\mu T^{-1})^{-1}\cdot T^{-1} \;=\; \Big(1+\mu T^{-1}+(\mu T^{-1})^2+\ldots\Big)\cdot T^{-1}$$

The geometric series converges in operator norm for $|\mu T^{-1}|_{\text{op}} < 1$, that is, for $|\mu| < |T^{-1}|_{\text{op}}^{-1}$. Having found the obvious candidate for an inverse,

$$(1 - \mu T^{-1}) \cdot \Big(1 + \mu T^{-1} + (\mu T^{-1})^2 + \ldots\Big) \;=\; 1$$

and

$$(T - \mu) \cdot \Big(1 + \mu T^{-1} + (\mu T^{-1})^2 + \ldots\Big) \cdot T^{-1} \;=\; 1$$

so there is a continuous linear operator $(T - \mu)^{-1}$, and $\mu \notin \sigma(T)$. Having already proven that $\sigma(T)$ is *bounded*, it is *compact*. ///

[1.4] Proposition: The spectrum $\sigma(T)$ of a continuous linear operator on a Hilbert space $V \neq \{0\}$ is *non-empty.*

Proof: The argument reduces the issue to Liouville's theorem from complex analysis, that a *bounded* entire (holomorphic) function is *constant.* Further, an entire function that goes to 0 at ∞ is identically 0.

Suppose the resolvent $R_\lambda = (T - \lambda)^{-1}$ is a continuous linear operator for all $\lambda \in \mathbb{C}$. The operator norm is readily estimated for large λ:

$$|R_\lambda|_{\text{op}} \;=\; |\lambda|^{-1} \cdot \Big|1 + \frac{T}{\lambda} + \big(\frac{T}{\lambda}\big)^2 + \ldots\Big|_{\text{op}}$$

$$\leq\; |\lambda|^{-1} \cdot \Big(1 + \big|\frac{T}{\lambda}\big|_{\text{op}} + \big|\frac{T}{\lambda}\big|_{\text{op}}^2 + \ldots\Big) \;=\; \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{|T|_{\text{op}}}{|\lambda|}}$$

This goes to 0 as $|\lambda| \to \infty$. *Hilbert's identity* asserts the complex differentiability as operator-valued function:

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} \;=\; R_\lambda \cdot \frac{(T - \mu) - (T - \lambda)}{\lambda - \mu} \cdot R_\mu \;=\; R_\lambda \cdot R_\mu \;\longrightarrow\; R_\lambda^2 \qquad (\text{as } \mu \to \lambda)$$

since $\mu \to R_\mu$ is continuous for large μ, by the same identity:

$$|R_\lambda - R_\mu|_{\text{op}} \;\leq\; |\lambda - \mu| \cdot |R_\mu \cdot R_\lambda|_{\text{op}}$$

Thus, the scalar-valued functions $\lambda \to \langle R_\lambda v, w\rangle$ for $v, w \in V$ are complex-differentiable, and satisfy

$$|\langle R_\lambda v, w\rangle| \;\leq\; |R_\lambda v| \cdot |w| \;\leq\; |R_\lambda|_{\text{op}} \cdot |v| \cdot |w| \;\leq\; \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{|T|_{\text{op}}}{|\lambda|}} \cdot |v| \cdot |w|$$

By Liouville, $\langle R_\lambda v, w\rangle = 0$ for all $v, w \in V$, which is impossible. Thus, the spectrum is not empty. ///

[1.5] Proposition: The entire spectrum, both point-spectrum and continuous-spectrum, of a *self-adjoint* operator is a non-empty, compact subset of $\mathbb{R}$. The entire spectrum of a *unitary* operator is a non-empty, compact subset of the unit circle.

Proof: For self-adjoint T, we claim that the imaginary part of $\langle (T - \mu)v, v\rangle$ is at least $\langle v, v\rangle$ times the imaginary part of μ. Indeed, $\langle Tv, v\rangle$ is *real*, since

$$\langle Tv, v\rangle \;=\; \langle v, T^*v\rangle \;=\; \langle v, Tv\rangle \;=\; \overline{\langle Tv, v\rangle}$$

so

$$\langle (T - \mu)v, v\rangle \;=\; \langle Tv, v\rangle - \mu \cdot \langle v, v\rangle$$

and

$$|\mathrm{Im}\,\langle (T-\mu)v, v\rangle| \;=\; |\mathrm{Im}\,\mu|\cdot\langle v,v\rangle$$

and by Cauchy-Schwarz-Bunyakowsky

$$|(T-\mu)v|\cdot|v| \;\ge\; |\langle (T-\mu)v, v\rangle| \;\ge\; |\mathrm{Im}\,\mu|\cdot\langle v,v\rangle \;=\; |\mathrm{Im}\,\mu|\cdot|v|^2$$

Dividing by $|v|$,

$$|(T-\mu)v| \;\ge\; |\mathrm{Im}\,\mu|\cdot|v|$$

This inequality shows more than the injectivity of $T-\mu$. Namely, the inequality gives a bound on the operator norm of the inverse $(T-\mu)^{-1}$ defined on the image of $T-\mu$. The image is *dense* since μ is not an eigenvalue and there is no residual spectrum for normal operators T. Thus, the inverse extends by continuity to a continuous linear map defined on the whole Hilbert space. Thus, $T-\mu$ has a continuous linear inverse, and μ is not in the spectrum of T.

For T unitary, $|Tv|=|v|$ for all v implies $T|_{\mathrm{op}}=1$. Thus, $\sigma(T)$ is contained in the unit disk, by the general bound on spectra in terms of operator norms. From $(T-\lambda)^*=T^*-\overline{\lambda}$, the spectrum of T^* is obtained by complex-conjugating the spectrum of T. Thus, for unitary T, the spectrum of $T^{-1}=T^*$ is also contained in the unit disk. At the same time, the natural

$$T-\lambda \;=\; -T\cdot(T^{-1}-\lambda^{-1})\cdot\lambda$$

gives

$$(T-\lambda)^{-1} \;=\; -\lambda^{-1}\cdot(T^{-1}-\lambda^{-1})^{-1}\cdot T^{-1}$$

so $\lambda^{-1}\in\sigma(T^{-1})$ exactly when $\lambda\in\sigma(T)$. Thus, the spectra of both T and $T^*=T^{-1}$ are inside the unit circle. ///

---

2. *Positive examples*

This section gives non-pathological examples. Let ℓ^2 be the usual space of square-summable sequences $(a_1,a_2,\ldots)$, with standard orthonormal basis

$$e_j \;=\; \underbrace{(0,\ldots,0,1,0,\ldots)}_{1\text{ at }jth\text{ position}}$$

[2.1] Multiplication operators with specified eigenvalues Given a countable, bounded list of complex numbers λ_j, the operator $T:\ell^2\to\ell^2$ by

$$T \;:\; (a_1,a_2,\ldots)\longrightarrow(\lambda_1\cdot a_1,\lambda_2\cdot a_2,\ldots)$$

has λ_j-eigenvector the standard basis element e_j. Clearly

$$T^* \;:\; (a_1,a_2,a_3,\ldots)\longrightarrow(\overline{\lambda}_1\cdot a_1,\overline{\lambda}_2\cdot a_2,\overline{\lambda}_3\cdot a_3,\ldots)$$

so T is *normal*, in the sense that $TT^*=T^*T$. To see that there are no *other* eigenvalues, suppose $Tv=\mu\cdot v$ with μ not among the λ_j. Then

$$\mu\cdot\langle v,e_j\rangle \;=\; \langle Tv,e_j\rangle \;=\; \langle v,T^*e_j\rangle \;=\; \langle v,\overline{\lambda}_je_j\rangle \;=\; \lambda_j\cdot\langle v,e_j\rangle$$

Thus, $(\mu-\lambda_j)\cdot\langle v,e_j\rangle=0$, and $\langle v,e_j\rangle=0$ for all j. Since e_j form an orthonormal basis, $v=0$. ///

[2.2] Every compact subset of $\mathbb{C}$ is the spectrum of an operator Grant for the moment a countable dense subset $\{\lambda_j\}$ of a non-empty compact subset[1] C of $\mathbb{C}$, and as above let

$$T : (a_1, a_2, a_3, \ldots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \lambda_3 \cdot a_3, \ldots)$$

We saw that there are no further eigenvalues. Since spectra are *closed*, the closure C of $\{\lambda_j\}$ is *contained* in $\sigma(T)$.

It remains to show that the continuous spectrum is no larger than the closure C of the eigenvalues, *in this example*. That is, for $\mu \notin C$, exhibit a continuous linear $(T-\mu)^{-1}$.

For $\mu \notin C$, there is a uniform lower bound $0 < \delta \le |\mu - \lambda_j|$. That is, $\sup_j |\mu - \lambda_j|^{-1} \le \delta^{-1}$. Thus, the naturally suggested map

$$(a_1, a_2, \ldots) \longrightarrow \Big((\lambda_1 - \mu)^{-1} \cdot a_1, \, (\lambda_2 - \mu)^{-1} \cdot a_2, \, \ldots\Big)$$

is a bounded linear map, and gives $(T-\mu)^{-1}$.

[2.3] Two-sided shift has no eigenvalues Let V be the Hilbert space of *two-sided* sequences $(\ldots, a_{-1}, a_0, a_1, \ldots)$ with natural inner product

$$\langle (\ldots, a_{-1}, a_0, a_1, \ldots), \, (\ldots, b_{-1}, b_0, b_1, \ldots) \rangle \;=\; \ldots + a_{-1} b_{-1} + a_0 b_0 + a_1 b_1 + \ldots$$

The right and left *two-sided* shift operators are

$$(R \cdot a)_n \;=\; a_{n-1} \qquad\qquad (L \cdot a)_n \;=\; a_{n+1}$$

These operators are mutual adjoints, mutual inverses, so are unitary. Being unitary, their operator norms are 1, so their spectra are non-empty compact subsets of the unit circle.

They have no eigenvalues: indeed, for $Rv = \lambda \cdot v$, if there is any index n with $v_n \neq 0$, then the relation $Rv = \lambda \cdot v$ gives $v_{n+k+1} = \lambda \cdot v_{n+k}$ for $k = 0, 1, 2, \ldots$. Since $|\lambda| = 1$, such a vector is not in ℓ^2.

Nevertheless, we claim that $\lambda \in \sigma(L)$ for every λ with $|\lambda| = 1$, and similarly for R. Indeed, for λ *not* in the spectrum, there is a continuous linear operator $(L-\lambda)^{-1}$, so $|(L-\lambda)v| \ge \delta \cdot |v|$ for some $\delta > 0$. It is easy to make *approximate* eigenvectors for L for any $|\lambda| = 1$: let

$$v^{(\ell)} \;=\; (\ldots, 0, \ldots, 0, 1, \lambda, \lambda^2, \lambda^3, \ldots, \lambda^\ell, 0, 0, \ldots)$$

Obviously it doesn't matter where the non-zero entries begin. From

$$(L-\lambda)v^{(\ell)} \;=\; (\ldots, 0, \ldots, 0, 1, 0, \ldots, 0, \lambda^{\ell+1}, 0, 0, \ldots)$$

$|(L-\lambda)v^{(\ell)}| = \sqrt{1+1}$, while $|v^{(\ell)}| = \sqrt{\ell+1}$. Thus, $|(L-\lambda)v^{(\ell)}|/|v^{(\ell)}| \longrightarrow 0$, and there can be no $(L-\lambda)^{-1}$. Thus, every λ on the unit circle is in $\sigma(R)$.

[2.4] Compact multiplication operators on ℓ^2 For a sequence of complex numbers $\lambda_n \to 0$, we claim that the multiplication operator

$$T : (a_1, a_2, \ldots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \ldots)$$

---

[1] To make a countable dense subset of C, for $n = 1, 2, \ldots$ cover C by finitely-many disks of radius $1/n$, each meeting C, and in each choose a point of C. The union over $n = 1, 2, \ldots$ of these finite sets is countable and dense in C.

is *compact*. We already showed that it has eigenvalues exactly $\lambda_1, \lambda_2, \ldots$, and spectrum the *closure* of $\{\lambda_j\}$. Thus, the spectrum includes 0, but 0 is an *eigenvalue* only when it appears among the λ_j, which may range from 0 times to infinitely-many times.

To prove that the operator is compact, we prove that the image of the unit ball is pre-compact, by showing that it is *totally bounded*. Given $\varepsilon > 0$, take k such that $|\lambda_i| < \varepsilon$ for $i > k$. The ball in $\mathbb{C}^k$ of radius $\max\{|\lambda_j| : j \leq k\}$ is precompact, so has a finite cover by ε-balls, centered at points $v^1, \ldots, v^N$. For $v = (v_1, v_2, \ldots)$ with $|v| \leq 1$,

$$Tv \;=\; (\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_k v_k, 0, 0, \ldots) + (0, \ldots, 0, \lambda_{k+1} v_{k+1}, \lambda_{k+2} v_{k+2}, \ldots)$$

With v^j the closest of the $v^1, \ldots, v^N$ to $(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_k v_k, 0, 0, \ldots)$,

$$|Tv - v^j| \;<\; \varepsilon + |(0, \ldots, 0, \lambda_{k+1} v_{k+1}, \lambda_{k+2} v_{k+2}, \ldots)| \;<\; \varepsilon + \varepsilon \cdot |(0, \ldots, 0, v_{k+1}, v_{k+2}, \ldots)| \;\leq\; \varepsilon + \varepsilon \cdot |v| \;\leq\; 2\varepsilon$$

Thus, the image of the unit ball under T is covered by finitely-many 2ε-balls. ///

[2.5] Multiplication operators on $L^2[a,b]$ For $\varphi \in C^o[a,b]$, we claim that the multiplication operator

$$M_\varphi \;:\; L^2[a,b] \longrightarrow L^2[a,b]$$

by

$$M_\varphi f(x) \;=\; \varphi(x) \cdot f(x)$$

is *normal*, and has spectrum the image $\varphi[a,b]$ of φ. The eigenvalues are λ such that $\varphi(x) = \lambda$ on a subset of $[a,b]$ of positive measure. The normality is clear, so, beyond eigenvalues, we need only examine continuous spectrum, not residual.

On one hand, if $\varphi(x) = \lambda$ on a set of positive measure, there is an infinite-dimensional sub-space of $L^2[0,1]$ of functions supported there, and all these are eigenvectors. On the other hand, if $f \neq 0$ in $L^2[0,1]$ and $\varphi(x) \cdot f(x) = \lambda \cdot f(x)$, even if f is altered on a set of measure 0, it must be that $\varphi(x) = \lambda$ on a set of positive measure.

To understand the continuous spectrum, for $\varphi(x_o) = \lambda$ make *approximate eigenvectors* by taking L^2 functions f supported on $[x_o - \delta, x_o + \delta]$, where $\delta > 0$ is small enough so that $|\varphi(x) - \varphi(x_o)| < \varepsilon$ for $|x - x_o| < \delta$. Then

$$|(M_\varphi - \lambda) f|^2_{L^2} \;=\; \int |\varphi(x) - \lambda|^2 \cdot |f(x)|^2 \, dx \;\leq\; \varepsilon^2 \cdot |f|^2_{L^2}$$

Thus, $\inf_{f \neq 0} |(M_\varphi - \lambda) f|_{L^2} / |f|_{L^2} = 0$, so $M_\varphi - \lambda$ is not invertible. If λ is not an eigenvalue, it is in the continuous spectrum. On the other hand, if $\varphi(x) \neq \lambda$, then there is some $\delta > 0$ such that $|\varphi(x) - \lambda| \geq \delta$ for all $x \in [0,1]$, by the compactness of $[0,1]$. Then

$$|(M_\varphi - \lambda) f|^2_{L^2} \;=\; \int_0^1 |\varphi(x) - \lambda|^2 \cdot |f(x)|^2 \, dx \;\geq\; \int_0^1 \delta^2 \cdot |f(x)|^2 \, dx \;=\; \delta^2 \cdot |f|^2_{L^2}$$

Thus, there is a continuous inverse $(M_\varphi - \lambda)$, and λ is *not* in the spectrum.

---

3. *Simplest Rellich compactness lemma*

One characterization of the s^{th} Levi-Sobolev space of functions $H^s(A)$ on a product $A = (S^1)^{\times n}$ of circles $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is as the closure of the function space of *finite* Fourier series with respect to the Levi-Sobolev norm (squared)

$$\Big| \sum_{\xi \in \mathbb{Z}^n} c_\xi \, e^{i\xi \cdot x} \Big|^2_{H^s} \;=\; \sum_{\xi \in \mathbb{Z}^n} |c_\xi|^2 \cdot (1 + |\xi|^2)^s \qquad (s \in \mathbb{R}, \text{ on finite Fourier series})$$

The standard orthonormal basis for $H^s(A)$ is

$$\frac{1}{(2\pi)^{n/2}} \cdot \frac{e^{i\xi\cdot x}}{(1+|\xi|^2)^{s/2}} \qquad \text{(with } \xi \in \mathbb{Z}^n\text{)}$$

By the Plancherel theorem, the map from $L^2(\mathbb{Z}^n)$ (with counting measure) to $L^2(A)$ by

$$\{c_\xi : \xi \in \mathbb{Z}^n\} \longrightarrow \frac{1}{(2\pi)^{n/2}} \sum_{\xi\in\mathbb{Z}^n} c_\xi \, \frac{e^{i\xi\cdot x}}{(1+|\xi|^2)^{s/2}}$$

is an isometric isomorphism.

For $s > t$, there is a continuous inclusion $H^s(A) \to H^t(A)$. In terms of these orthonormal bases, there is a commutative diagram

$$\begin{array}{ccc} L^2(A) & \xrightarrow{T} & L^2(A) \\ \approx\downarrow & & \downarrow\approx \\ H^s(A) & \xrightarrow[\text{inc}]{} & H^t(A) \end{array}$$

given by

$$\begin{array}{ccc} \{c_\xi\} & \xrightarrow{T} & \left\{(1+|\xi|^2)^{\frac{t-s}{2}} \cdot c_\xi\right\} \\ \approx\downarrow & & \downarrow\approx \\ \frac{1}{(2\pi)^{n/2}} \sum_\xi c_\xi \, \frac{e^{i\xi\cdot x}}{(1+|\xi|^2)^{s/2}} & \xrightarrow[\text{inc}]{} & \frac{1}{(2\pi)^{n/2}} \sum_\xi (1+|\xi|^2)^{\frac{t-s}{2}} \cdot c_\xi \, \frac{e^{i\xi\cdot x}}{(1+|\xi|^2)^{t/2}} \end{array}$$

Since $s > t$, the number $\lambda_\xi = (1+|\xi|^2)^{\frac{t-s}{2}}$ are bounded by 1, and have unique limit point 0. In particular, $T : L^2(A) \to L^2(A)$ is *compact*.

Thus, we have the simplest instance of *Rellich's compactness lemma*: the inclusion $H^s(A) \to H^t(A)$ is *compact* for $s > t$.

---

4. *Cautionary examples: non-normal operators*

[4.1] Shift operators on one-sided ℓ^2 We claim the following: The right-shift

$$R : (a_1, a_2, \ldots) \longrightarrow (0, a_1, a_2, \ldots)$$

and the left-shift

$$L : (a_1, a_2, a_3, \ldots) \longrightarrow (a_2, \ldots)$$

are mutual adjoints. These operators are not normal, since $L \circ R = 1_{\ell^2}$ but

$$R \circ L : (a_1, a_2, \ldots) \longrightarrow (0, a_2, \ldots)$$

The eigenvalues of the left-shift L are all complex numbers in the open unit disk in $\mathbb{C}$. In particular, there is a *continuum* of eigenvalues and eigenvectors, so they *cannot be mutually orthogonal*. The spectrum $\sigma(L)$ is the closed unit disk.

The right-shift R has *no* eigenvalues, has continuous spectrum the unit circle, and residual spectrum the open unit disk with 0 removed.

Indeed, suppose

$$(0, a_1, a_2, \ldots) \;=\; R(a_1, a_2, \ldots) \;=\; \lambda \cdot (a_1, a_2, \ldots)$$

With n the lowest index such that $a_n \neq 0$, the n^{th} component in the eigenvector relation gives $0 = a_{n-1} = \lambda \cdot a_n$, so $\lambda = 0$. Then, the $(n+1)^{th}$ component gives $a_n = \lambda \cdot a_{n+1} = 0$, contradiction. This proves that R has *no* eigenvalues.

Oppositely, for $|\lambda| < 1$,

$$L(1, \lambda, \lambda^2, \ldots) \;=\; (\lambda, \lambda^2, \ldots) \;=\; \lambda \cdot (1, \lambda, \lambda^2, \ldots)$$

so every such λ is an eigenvector for L. On the other hand, for $|\lambda| = 1$, in an eigenvector relation

$$(a_2, \ldots) \;=\; L(a_1, a_2, \ldots) \;=\; \lambda \cdot (a_1, a_2, \ldots)$$

let n be the smallest index n with $a_n \neq 0$. Then $a_{n+1} = \lambda \cdot a_n$, $a_{n+2} = \lambda \cdot a_{n+1}$, ..., so

$$(a_1, a_2, \ldots) \;=\; (0, \ldots, 0, a_n, \lambda a_n, \lambda^2 a_n, \ldots)$$

But this is not in ℓ^2 for $|\lambda| = 1$ and $a_n \neq 0$, so λ on the unit circle is *not* an eigenvalue.

For $|\lambda| = 1$, we can make *approximate* λ*-eigenvectors* for L by

$$v^{[N]} \;=\; (1, \lambda, \lambda^2, \ldots, \lambda^N, 0, 0, \ldots)$$

since

$$(L - \lambda)\, v^{[N]} \;=\; (\lambda, \lambda^2, \ldots, \lambda^N, 0, 0, 0, \ldots) - \lambda \cdot (1, \lambda, \lambda^2, \ldots, \lambda^N, 0, 0, \ldots) \;=\; (0, 0, \ldots, 0, 0, \lambda^{N+1}, 0, 0, \ldots)$$

Since

$$\frac{|(L - \lambda)v^{[N]}|}{|v^{[N]}|} \;=\; \frac{|\lambda|^{N+1}}{(1 + |\lambda|^2 + \ldots + |\lambda|^{2N})^{1/2}} \;=\; \frac{1}{\sqrt{N+1}} \;\longrightarrow\; 0$$

there can be no continuous $(L - \lambda)^{-1}$. Thus, λ on the unit circle is in the spectrum, but not in the point spectrum.

That the unit circle is in the spectrum also follows from the observation above that all λ with $|\lambda| < 1$ are eigenvalues, and the fact that the spectrum is *closed*.

The spectrum of L is bounded by the operator norm $|L|_{\text{op}}$, and $|L|_{\text{op}}$ is visibly 1, so is nothing *else* in the spectrum.

To see that the unit circle is the *continuous* spectrum of L, as opposed to *residual*, we show that $L - \lambda$ has dense image for $|\lambda| = 1$. Indeed, for w such that, for all $v \in \ell^2$,

$$0 \;=\; \langle (L - \lambda)v, w \rangle \;=\; \langle v, (L^* - \overline{\lambda})w \rangle \;=\; \langle v, (R - \overline{\lambda})w \rangle$$

we would have $(R - \overline{\lambda})w = 0$. However, we have seen that R has no eigenvalues. Thus, $L - \lambda$ always has dense image, and the unit circle is continuous spectrum for L.

Reversing that discussion, every λ with $|\lambda| < 1$ is in the residual spectrum of R, because such λ is not an eigenvalue, and $R - \lambda$ does *not* have dense image: for w a $\overline{\lambda}$-eigenvector for L,

$$\langle (R - \lambda)v, w \rangle \;=\; \langle v, (R^* - \overline{\lambda})w \rangle \;=\; \langle v, (L - \overline{\lambda})w \rangle \;=\; \langle v, 0 \rangle \;=\; 0$$

That is, the image $(R - \lambda)\ell^2$ is in the orthogonal complement to the eigenvector w. The same computation shows that the unit circle is *continuous* spectrum for R, because it is *not* eigenvalues for L.

[4.2] Volterra operator We will show that the Volterra operator $Vf(x) = \int_0^x f(t)\,dt$ on $L^2[0,1]$ is *compact*, but not self-adjoint, that its spectrum is $\{0\}$, and that it has no eigenvalues.

A relation $Tf = \lambda \cdot f$ for $f \in L^2$ and $\lambda \neq 0$ implies f is *continuous*:

$$|\lambda| \cdot |f(x+h) - f(x)| \;=\; |Tf(x+h) - Tf(x)| \;\leq\; \int_x^{x+h} 1 \cdot |f(t)|\,dt \;=\; \leq\; |h|^{\frac{1}{2}} \cdot |f|_{L^2}$$

The fundamental theorem of calculus would imply f is continuously differentiable and $\lambda \cdot f' = (Tf)' = f$. Thus, f would be a constant multiple of $e^{x/\lambda}$, by the mean value theorem. However, by Cauchy-Schwarz-Bunyakowsky, for a λ-eigenfunction

$$|\lambda| \cdot |f(x)| \;\leq\; |x|^{\frac{1}{2}} \cdot |f|_{L^2}$$

No non-zero multiple of the exponential satisfies this. Thus, there are no eigenvectors for *non-zero* eigenvalues.

For $f \in L^2[0,1]$ and $Tf = 0 \in L^2[0,1]$, Tf is almost everywhere 0. Since $x \to Tg(x)$ is unavoidably *continuous*, $Tf(x)$ is 0 for all x. Thus, for all x, y in the interval,

$$0 \;=\; 0 - 0 \;=\; Tf(y) - Tf(x) \;=\; \int_x^y f(t)\,dt$$

That is, $x \to Tf(x)$ is orthogonal in $L^2[0,1]$ to all characteristic functions of intervals. Finite linear combinations of these are dense in $C^o[0,1]$ in the L^2 topology, and $C^o[0,1]$ is dense in $L^2[0,1]$. Thus $f = 0$, and there are no eigenvectors for the Volterra operator.

To see that T is *compact*, rewrite it as being given by an *integral kernel* $K(x,y)$:

$$Tf(x) \;=\; \int_0^x f(y)\,dy \;=\; \int_0^1 K(x,y)\,f(y)\,dy \qquad\qquad \left(\text{with } K(x,y) = \begin{cases} 0 & (\text{for } 0 \leq y < x) \\ 1 & (\text{for } x < y \leq 1) \end{cases}\right)$$

Thus, T is Hilbert-Schmidt, and compact. The adjoint T^* is given by the integral kernel $K^*(x,y) = \overline{K(y,x)}$, visibly different from $K(x,y)$, so T is *not* self-adjoint.

To see that the spectrum is *at most* $\{0\}$, show that the *spectral radius* is 0:

$$T^n f(x) \;=\; \int_0^x \int_0^{x_{n-1}} \ldots \int_0^{x_2} \int_0^{x_1} f(t)\,dt\,dx_1 \ldots dx_{n-1} \;=\; \int_0^x f(t) \Big(\int_t^x \int_t^{x_{n-1}} \ldots \int_t^{x_2} dx_1 \ldots dx_{n-1} \Big)\,dt$$

$$= \int_0^x f(t) \cdot \frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

From this, $|T^n|_{\text{op}} \leq \frac{1}{n!}$, and

$$\log \lim_{2n} \Big(\frac{1}{(2n)!}\Big)^{1/2n} \;=\; -\lim_{2n} \frac{1}{2n} \cdot \log(2n)! \;=\; -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq 2n} \log k$$

$$= -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq n} (\log k + \log(2n-k+1)) \;\leq\; -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq \frac{n}{k}} (\log k + \log(2n-k+1))$$

$$\leq\; -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq \frac{n}{k}} \log 2n \;=\; -\lim_{2n} \frac{\log 2n}{2} \;=\; -\infty$$

since $k(2n-k) \geq 2n$ for $1 \leq k \leq n$, noting the sign. That is, $\lim_n |T^n|_{\text{op}}^{1/n} = 0$, so the spectral radius is 0. Since the spectrum is non-empty, it must be exactly $\{0\}$.
