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## 08c. Nuclear spaces, Schwartz' kernel theorem

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Hilbert-Schmidt operators  $T : L^2(X) \rightarrow L^2(Y)$  are usefully described in terms of their *Schwartz kernels*  $K(x, y)$ , such that

$$Tf(y) = \int_Y K(x, y) f(x) dx$$

Unfortunately, not all continuous linear maps  $T : L^2(X) \rightarrow L^2(Y)$  have Schwartz kernels, unless one or the other of the two spaces is *finite-dimensional*.

Sufficiently enlarging the class of possible  $K(x, y)$  turns out to require a family of topological vector spaces with *tensor products*.<sup>[1]</sup> The connection between integral/Schwartz kernels and tensor products is suggested by the prototypical Cartan-Eilenberg *adjunction*, for example for  $k$ -vectorspaces without topologies: with the usual tensor product of vector spaces,

$$\mathrm{Hom}_k(A, \mathrm{Hom}_k(B, C)) \approx \mathrm{Hom}(A \otimes_k B, C) \quad (\text{by } \varphi \rightarrow (a \otimes b \rightarrow \varphi(a)(b)))$$

The special case  $C = k$  gives

$$\mathrm{Hom}_k(A, B^*) \approx \mathrm{Hom}(A \otimes_k B, k) = (A \otimes_k B)^* \quad (k\text{-vectorspaces } A, B, C)$$

That is, maps from  $A$  to  $B^*$  are given by *integral kernels* in  $(A \otimes B)^*$ . However, the validity of this adjunction depends on existence of a genuine tensor product. We recall in an appendix the demonstration that infinite-dimensional Hilbert spaces do *not* have tensor products. Also, we must specify the topology on the duals  $B^*$  and  $(A \otimes B)^*$ . The strongest conclusion gives these the *strong* topology, as colimit of Hilbert-space topologies on the duals of Hilbert spaces.

Countable projective limits of Hilbert spaces with transition maps Hilbert-Schmidt constitute the simplest class of *nuclear spaces*: they admit *tensor products*. The simplest example of such a space is the Levi-Sobolev space  $H^\infty(\mathbb{T}^n)$  on a product  $\mathbb{T}^n$  of circles  $\mathbb{T} = S^1$ , where the simplest Rellich-Kondrachev compactness lemma is easily refined to prove the requisite Hilbert-Schmidt property.

The main corollary of existence of tensor products of nuclear spaces is *Schwartz' Kernel Theorem*, which provides a framework for later discussion of *pseudo-differential operators*, for example.

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[1] A categorically genuine *tensor product* of topological vector spaces  $V, W$  would be a topological vector space  $X$  and continuous bilinear map  $j : V \times W \rightarrow X$  such that, for every continuous bilinear  $V \times W \rightarrow Y$  to a topological vector space  $Y$ , there is a unique continuous linear  $X \rightarrow Y$  fitting into the commutative diagram

$$\begin{array}{ccc} & X & \\ & \uparrow & \searrow \\ j & & \\ \uparrow & & \\ V \times W & \longrightarrow & Y \end{array}$$

## 1. Hilbert-Schmidt operators

### [1.1] Prototype: integral operators

For  $K(x, y)$  in  $C^o([a, b] \times [a, b])$ , define  $T : L^2[a, b] \rightarrow L^2[a, b]$  by

$$Tf(y) = \int_a^b K(x, y) f(x) dx$$

The function  $K$  is the *integral kernel*, or *Schwartz kernel* of  $T$ . Approximating  $K$  by *finite* linear combinations of 0-or-1-valued functions shows  $T$  is a uniform operator norm limit of finite-rank operators, so is *compact*. The *Hilbert-Schmidt* operators include such operators, where the integral kernel  $K(x, y)$  is allowed to be in  $L^2([a, b] \times [a, b])$ .

### [1.2] Hilbert-Schmidt norm on $V \otimes_{\text{alg}} W$

In the category of Hilbert spaces and continuous linear maps, there is *no* tensor product in the categorical sense, as demonstrated in an appendix.

*Without* claiming anything about genuine tensor products in any category of topological vector spaces, the *algebraic* tensor product  $X \otimes_{\text{alg}} Y$  of two Hilbert spaces has a hermitian inner product  $\langle \cdot, \cdot \rangle_{\text{HS}}$  determined by

$$\langle x \otimes y, x' \otimes y' \rangle_{\text{HS}} = \langle x, x' \rangle \langle y, y' \rangle$$

Let  $X \otimes_{\text{HS}} Y$  be the completion with respect to the corresponding norm  $|v|_{\text{HS}} = \langle v, v \rangle_{\text{HS}}^{1/2}$

$$X \otimes_{\text{HS}} Y = |\cdot|_{\text{HS}}\text{-completion of } X \otimes_{\text{alg}} Y$$

This completion is a Hilbert space.

### [1.3] Hilbert-Schmidt operators

For Hilbert spaces  $V, W$  the finite-rank<sup>[2]</sup> continuous linear maps  $T : V \rightarrow W$  can be identified with the algebraic tensor product  $V^* \otimes_{\text{alg}} W$ , by<sup>[3]</sup>

$$(\lambda \otimes w)(v) = \lambda(v) \cdot w$$

The space of *Hilbert-Schmidt operators*  $V \rightarrow W$  is the completion of the space  $V^* \otimes_{\text{alg}} W$  of finite-rank operators, with respect to the *Hilbert-Schmidt norm*  $|\cdot|_{\text{HS}}$  on  $V^* \otimes_{\text{alg}} W$ . For example,

$$\begin{aligned} |\lambda \otimes w + \lambda' \otimes w'|_{\text{HS}}^2 &= \langle \lambda \otimes w + \lambda' \otimes w', \lambda \otimes w + \lambda' \otimes w' \rangle \\ &= \langle \lambda \otimes w, \lambda \otimes w \rangle + \langle \lambda \otimes w, \lambda' \otimes w' \rangle + \langle \lambda' \otimes w', \lambda \otimes w \rangle + \langle \lambda' \otimes w', \lambda' \otimes w' \rangle \\ &= |\lambda|^2 |w|^2 + \langle \lambda, \lambda' \rangle \langle w, w' \rangle + \langle \lambda', \lambda \rangle \langle w', w \rangle + |\lambda'|^2 |w'|^2 \end{aligned}$$

[2] As usual a *finite-rank* linear map  $T : V \rightarrow W$  is one with finite-dimensional image.

[3] Proof of this identification: on one hand, a map coming from  $V^* \otimes_{\text{alg}} W$  is a *finite* sum  $\sum_i \lambda_i \otimes w_i$ , so certainly has finite-dimensional image. On the other hand, given  $T : V \rightarrow W$  with finite-dimensional image, take  $v_1, \dots, v_n$  be an orthonormal basis for the orthogonal complement  $(\ker T)^\perp$  of  $\ker T$ . Define  $\lambda_i \in V^*$  by  $\lambda_i(v) = \langle v, v_i \rangle$ . Then  $T \sim \sum_i \lambda_i \otimes Tv_i$  is in  $V^* \otimes W$ . The second part of the argument uses the completeness of  $V$ .

When  $\lambda \perp \lambda'$  or  $w \perp w'$ , the monomials  $\lambda \otimes w$  and  $\lambda' \otimes w'$  are orthogonal, and

$$|\lambda \otimes w + \lambda' \otimes w'|_{\text{HS}}^2 = |\lambda|^2|w|^2 + |\lambda'|^2|w'|^2$$

That is, the space  $\text{Hom}_{\text{HS}}(V, W)$  of Hilbert-Schmidt operators  $V \rightarrow W$  is the *closure* of the space of finite-rank maps  $V \rightarrow W$ , in the space of all continuous linear maps  $V \rightarrow W$ , under the Hilbert-Schmidt norm. By construction,  $\text{Hom}_{\text{HS}}(V, W)$  is a Hilbert space.

#### [1.4] Expressions for Hilbert-Schmidt norm, adjoints

The Hilbert-Schmidt norm of finite-rank  $T : V \rightarrow W$  can be computed from any choice of orthonormal basis  $v_i$  for  $V$ , by

$$|T|_{\text{HS}}^2 = \sum_i |Tv_i|^2 \quad (\text{at least for finite-rank } T)$$

Thus, taking a limit, the same formula computes the Hilbert-Schmidt norm of  $T$  known to be Hilbert-Schmidt. Similarly, for two Hilbert-Schmidt operators  $S, T : V \rightarrow W$ ,

$$\langle S, T \rangle_{\text{HS}} = \sum_i \langle Sv_i, Tv_i \rangle \quad (\text{for any orthonormal basis } v_i)$$

The Hilbert-Schmidt norm  $|\cdot|_{\text{HS}}$  dominates the *uniform operator norm*  $|\cdot|_{\text{op}}$ : given  $\varepsilon > 0$ , take  $|v_1| \leq 1$  with  $|Tv_1|^2 + \varepsilon > |T|_{\text{op}}^2$ . Choose  $v_2, v_3, \dots$  so that  $v_1, v_2, \dots$  is an orthonormal basis. Then

$$|T|_{\text{op}}^2 \leq |Tv_1|^2 + \varepsilon \leq \varepsilon + \sum_n |Tv_n|^2 = \varepsilon + |T|_{\text{HS}}^2$$

This holds for every  $\varepsilon > 0$ , so  $|T|_{\text{op}}^2 \leq |T|_{\text{HS}}^2$ . Thus, Hilbert-Schmidt limits are operator-norm limits, and Hilbert-Schmidt limits of finite-rank operators are *compact*.

*Adjoint*s  $T^* : W \rightarrow V$  of Hilbert-Schmidt operators  $T : V \rightarrow W$  are Hilbert-Schmidt, since for an orthonormal basis  $w_j$  of  $W$

$$\sum_i |Tv_i|^2 = \sum_{ij} |\langle Tv_i, w_j \rangle|^2 = \sum_{ij} |\langle v_i, T^*w_j \rangle|^2 = \sum_j |T^*w_j|^2$$

#### [1.5] Criterion for Hilbert-Schmidt operators

We claim that a continuous linear map  $T : V \rightarrow W$  with Hilbert space  $V$  is Hilbert-Schmidt if for some orthonormal basis  $v_i$  of  $V$

$$\sum_i |Tv_i|^2 < \infty$$

and then (as above) that sum computes  $|T|_{\text{HS}}^2$ . Indeed, given that inequality, letting  $\lambda_i(v) = \langle v, v_i \rangle$ ,  $T$  is Hilbert-Schmidt because it is the Hilbert-Schmidt limit of the finite-rank operators

$$T_n = \sum_{i=1}^n \lambda_i \otimes Tv_i$$

#### [1.6] Composition of Hilbert-Schmidt operators with continuous operators

Post-composing: for Hilbert-Schmidt  $T : V \rightarrow W$  and continuous  $S : W \rightarrow X$ , the composite  $S \circ T : V \rightarrow X$  is Hilbert-Schmidt, because for an orthonormal basis  $v_i$  of  $V$ ,

$$\sum_i |S \circ Tv_i|^2 \leq \sum_i |S|_{\text{op}}^2 \cdot |Tv_i|^2 = |S|_{\text{op}} \cdot |T|_{\text{HS}}^2 \quad (\text{with operator norm } |S|_{\text{op}} = \sup_{|v| \leq 1} |Sv|)$$

Pre-composing: for continuous  $S : X \rightarrow V$  with Hilbert  $X$  and orthonormal basis  $x_j$  of  $X$ , since adjoints of Hilbert-Schmidt are Hilbert-Schmidt,

$$T \circ S = (S^* \circ T^*)^* = (\text{Hilbert-Schmidt})^* = \text{Hilbert-Schmidt}$$

## 2. Simplest nuclear Fréchet spaces

Roughly, the intention of *nuclear spaces* is that they should admit genuine *tensor products*, aiming at a general Schwartz Kernel Theorem.

For the moment, we consider a more accessible sub-class of nuclear spaces, sufficient for the Schwartz Kernel Theorem for Levi-Sobolev spaces  $H^s(\mathbb{T}^n)$  below: countable projective limits of Hilbert spaces with Hilbert-Schmidt transition maps. Thus, they are also Fréchet, so are among *nuclear Fréchet spaces*.

### [2.1] $V \otimes_{\text{HS}} W$ is not a categorical tensor product

Again, the Hilbert space  $V \otimes_{\text{HS}} W$  is not a categorical tensor product of (infinite-dimensional) Hilbert spaces  $V, W$ . In particular, although the bilinear map  $V \times W \rightarrow V \otimes_{\text{HS}} W$  is continuous, there are (jointly) continuous  $\beta : V \times W \rightarrow X$  to Hilbert spaces  $H$  which do *not* factor through any continuous linear map  $B : V \otimes_{\text{HS}} W \rightarrow X$ .

The case  $W = V^*$  and  $X = \mathbb{C}$ , with  $\beta(v, \lambda) = \lambda(v)$  already illustrates this point, since not every Hilbert-Schmidt operator has a trace. That is, letting  $v_i$  be an orthonormal basis for  $V$  and  $\lambda_i(v) = \langle v, v_i \rangle$  an orthonormal basis for  $V^*$ , necessarily

$$B\left(\sum_{ij} c_{ij} v_i \otimes \lambda_j\right) = \sum_{ij} c_{ij} \beta(v_i, \lambda_j) = \sum_i c_{ii} \quad (???)$$

However,  $\sum_i \frac{1}{i} v_i \otimes \lambda_i$  is in  $V \otimes_{\text{HS}} V^*$ , but the alleged value of  $B$  is impossible. In effect, the obstacle is that there are Hilbert-Schmidt maps which are not of trace class.

### [2.2] Approaching tensor products and nuclear spaces

Let  $V, W, V_1, W_1$  be Hilbert spaces with Hilbert-Schmidt maps  $S : V_1 \rightarrow V$  and  $T : W_1 \rightarrow W$ . We claim that for any (jointly) continuous  $\beta : V \times W \rightarrow X$ , there is a unique continuous  $B : V_1 \otimes_{\text{HS}} W_1 \rightarrow X$  giving a commutative diagram

$$\begin{array}{ccccc}
 & & & & B \\
 & & & & \curvearrowright \\
 V_1 \otimes_{\text{HS}} W_1 & \longrightarrow & V \otimes_{\text{HS}} W & & \\
 \uparrow & & \uparrow & & \searrow \\
 V_1 \times W_1 & \xrightarrow{S \times T} & V \times W & \xrightarrow{\beta} & X
 \end{array}$$

In fact,  $B : V_1 \otimes_{\text{HS}} W_1 \rightarrow X$  is *Hilbert-Schmidt*. As the diagram suggests,  $V \otimes_{\text{HS}} W$  is bypassed, playing no role.

*Proof:* Once the assertion is formulated, the argument is the only thing it can be: The continuity of  $\beta$  gives a constant  $C$  such that  $|\beta(v, w)| \leq C \cdot |v| \cdot |w|$ , for all  $v \in V, w \in W$ . The Hilbert-Schmidt condition is that, for chosen orthonormal bases  $v_i$  of  $V_1$  and  $w_j$  of  $W_1$ ,

$$|S|_{\text{HS}}^2 = \sum_i |Sv_i|^2 < \infty \quad |T|_{\text{HS}}^2 = \sum_j |Tw_j|^2 < \infty$$

Thus,

$$|\beta(Sv, Tw)| \leq C \cdot |Sv| \cdot |Tw|$$

Squaring and summing over  $v_i$  and  $w_j$ ,

$$\sum_{ij} |\beta(Sv_i, Tw_j)|^2 \leq C \cdot \sum_{ij} |Sv_i|^2 \cdot |Tw_j|^2 = C \cdot |S|_{\text{HS}}^2 \cdot |T|_{\text{HS}}^2 < \infty$$

That is, with the obvious definition-attempt

$$B\left(\sum_{ij} c_{ij} v_i \otimes w_j\right) = \sum_{ij} c_{ij} \beta(Sv_i, Tw_j)$$

Cauchy-Schwarz-Bunyakowsky

$$\sum_{ij} |c_{ij} \beta(Sv_i, Tw_j)|^2 \leq \sum_{ij} |c_{ij}|^2 \cdot \sum_{ij} |\beta(Sv_i, Tw_j)|^2 \leq \sum_{ij} |c_{ij}|^2 \cdot \left(C \cdot |S|_{\text{HS}}^2 \cdot |T|_{\text{HS}}^2\right)$$

shows that  $B : V_1 \otimes W_1 \rightarrow X$  is Hilbert-Schmidt. ///

### [2.3] A class of nuclear Fréchet spaces

We take the basic *nuclear Fréchet space* to be a countable limit [4] of Hilbert spaces where the transition maps are *Hilbert-Schmidt*.

That is, for a countable collection of Hilbert spaces  $V_0, V_1, V_2, \dots$  with *Hilbert-Schmidt* maps  $\varphi_i : V_i \rightarrow V_{i-1}$ , the limit  $V = \lim_i V_i$  in the category of locally convex topological vector spaces is a *nuclear Fréchet space*. [5]

Let  $\mathfrak{C}$  be the category of Hilbert spaces enlarged to include limits.

[2.4] **Theorem:** *Nuclear Fréchet spaces admit tensor products* in  $\mathfrak{C}$ . That is, for nuclear spaces  $V = \lim_i V_i$  and  $W = \lim W_i$  there is a nuclear space  $V \otimes W$  and continuous bilinear  $V \times W \rightarrow V \otimes W$  such that, given a jointly continuous bilinear map  $\beta : V \times W \rightarrow X$  of nuclear spaces  $V, W$  to  $X \in \mathfrak{C}$ , there is a unique continuous linear map  $B : V \otimes W \rightarrow X$  giving a commutative diagram

$$\begin{array}{ccc} V \otimes W & & \\ \uparrow & \searrow B & \\ V \times W & \xrightarrow{\beta} & X \end{array}$$

In particular,  $V \otimes W \approx \lim_i V_i \otimes_{\text{HS}} W_i$ .

*Proof:* As will be seen at the end of this proof, the defining property of (projective) limits reduces to the case that  $X$  is itself a Hilbert space. Let  $\varphi_i : V_i \rightarrow V_{i-1}$  and  $\psi_i : W_i \rightarrow W_{i-1}$  be the transition maps. First, we claim that, for large-enough index  $i$ , the bilinear map  $\beta : V \times W \rightarrow X$  factors through  $V_i \times W_i$ . Indeed,

[4] Properly, the class of categorical *limits* includes *products* and other objects whose indexing sets are not necessarily *directed*. In that context, requiring that the index set be directed, a projective limit is a *directed* or *filtered* limit. Similarly, what we will call simply *colimits* are properly *filtered* or *directed* colimits.

[5] The new aspect is the nuclearity, not the Fréchet-ness: an arbitrary *countable* limit of Hilbert spaces is (provably) Fréchet, since an arbitrary countable limit of *Fréchet* spaces is Fréchet.



Thus,

$$C^\infty(\mathbb{T}^m) = H^{+\infty}(\mathbb{T}^m) = \lim_s H^s(\mathbb{T}^m) \approx \lim \left( \dots \rightarrow H^2(\mathbb{T}^m) \rightarrow H^1(\mathbb{T}^m) \rightarrow H^0(\mathbb{T}^m) \right)$$

We claim that

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \approx H^{+\infty}(\mathbb{T}^{m+n})$$

induced from the natural

$$(\varphi \otimes \psi)(x, y) = \varphi(x) \psi(y) \quad (\varphi \in H^{+\infty}(\mathbb{T}^m), \psi \in H^{+\infty}(\mathbb{T}^n), x \in \mathbb{T}^m, y \in \mathbb{T}^n)$$

Indeed, our construction of this tensor product is

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) = \lim_s \left( H^s(\mathbb{T}^m) \otimes_{\text{HS}} H^s(\mathbb{T}^n) \right)$$

The inequalities

$$(1 + |\xi|^2 + |\eta|^2)^2 \geq (1 + |\xi|^2)(1 + |\eta|^2) \geq 1 + |\xi|^2 + |\eta|^2 \quad (\text{for } \xi \in \mathbb{Z}^m, \eta \in \mathbb{Z}^n)$$

give

$$H^{2s}(\mathbb{T}^{m+n}) \subset H^s(\mathbb{T}^m) \otimes_{\text{HS}} H^s(\mathbb{T}^n) \subset H^s(\mathbb{T}^{m+n}) \quad (\text{for } s \geq 0)$$

The limit only depends on cofinal sublimits, so, indeed,

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \approx H^{+\infty}(\mathbb{T}^{m+n})$$

### 3. Schwartz Kernel Theorem for Levi-Sobolev spaces

Continue the example of Levi-Sobolev spaces on products  $\mathbb{T}^m$  of circles  $\mathbb{T}$ . The following is the simplest example of Schwartz' Kernel Theorem:

[3.1] Theorem: We have an *isomorphism*

$$\text{Hom}^o(H^\infty(\mathbb{T}^m), H^{-\infty}(\mathbb{T}^n)) \approx H^{-\infty}(\mathbb{T}^{m+n})$$

induced by

$$(f \rightarrow (F \rightarrow \Phi(f \otimes F))) \leftarrow \Phi \quad (\text{with } f \in H^\infty(\mathbb{T}^m), F \in H^\infty(\mathbb{T}^n), \Phi \in H^{-\infty}(\mathbb{T}^{m+n}))$$

The distribution  $\Phi \in H^{-\infty}(\mathbb{T}^{m+n})$  producing a given continuous map  $H^\infty(\mathbb{T}^m) \rightarrow H^{-\infty}(\mathbb{T}^n)$  is the *Schwartz kernel* of the map.

[3.2] Remark: The Hom-space  $\text{Hom}^o$  is *continuous* linear maps, so giving sense to the assertion requires a topology on the dual space  $H^{-\infty}(\mathbb{T}^n) = H^\infty(\mathbb{T}^n)^*$ . We give this space the *weak dual* topology.

*Proof:* Let  $X = H^\infty(\mathbb{T}^m)$  and  $Y = H^\infty(\mathbb{T}^n)$ , Given the existence of the categorical tensor product, established above, it suffices to show that the vector space

$$\text{Bil}^o(X \times Y, \mathbb{C})$$

of jointly continuous bilinear maps is linearly isomorphic to  $\text{Hom}(X, Y^*)$ , via the expected

$$\beta \rightarrow (x \rightarrow (y \rightarrow \beta(x, y))) \quad (\text{for } \beta \in \text{Bil}^o(X, Y), x \in X, \text{ and } y \in Y)$$

where  $Y^*$  is given the *strong dual* topology, namely, as colimit of Hilbert-space topologies on the duals  $H^{-s}(\mathbb{T}^n)$  with  $-s < 0$ . The issue is topological.

Given  $x \in X$ , bounded  $E \subset Y$ , and  $\varepsilon > 0$ , by joint continuity of  $\beta$ , there are neighborhoods  $M, N$  of 0 in  $X, Y$  such that

$$\beta(x + M, N) = \beta(x + M, N) - \beta(x, 0) \subset \varepsilon\text{-ball in } Y^*$$

Since  $E$  is bounded, there is  $t > 0$  such that  $tN \supset E$ . Then

$$\beta(x + m, e) - \beta(x, e) = \beta(m, e) \in \beta(M, E) \subset \beta(M, tN) \quad (\text{for } m \in M \text{ and } e \in E)$$

This suggests replacing  $M$  by  $t^{-1}M$ , so

$$\beta(x + m, e) - \beta(x, e) = \beta(t^{-1}M, E) \subset \beta(t^{-1}M, tN) \subset \varepsilon\text{-ball in } Y^* \quad (\text{for } m \in t^{-1}M \text{ and } e \in E)$$

That is,

$$\beta(x + m, -) - \beta(x, -) \in U_{E, \varepsilon} \quad (\text{for } m \in t^{-1}M)$$

This proves the continuity of the map  $X \rightarrow Y^*$  induced by  $\beta$ .

Conversely, given  $\varphi : X \rightarrow Y^*$ , put  $\beta(x, y) = \varphi(x)(y)$ . For fixed  $x$ ,  $\beta(x, -) = \varphi(x)$  is continuous, by hypothesis. For fixed  $y$ ,  $E = \{y\}$  is a bounded set in  $Y$ , so by the continuity of  $x \rightarrow \varphi(x)$ , for given  $x$  and  $\varepsilon > 0$  there is a neighborhood  $M$  of 0 in  $X$  so that  $\varphi(x + M) - \varphi(x) \subset U_{E, \varepsilon}$ . This proves that  $\beta(-, y)$  is continuous. Thus,  $\beta$  is *separately* continuous. An appendix shows that separately continuous bilinear functions on Hilbert spaces are jointly continuous. ///

## 4. Appendix: joint continuity of bilinear maps

Joint continuity of *separately* continuous bilinear maps on Hilbert spaces, is a corollary of Baire category:

**[4.1] Claim:** A bilinear map  $\beta : X \times Y \rightarrow Z$  on Hilbert spaces  $X, Y, Z$ , continuous in each variable *separately*, is *jointly* continuous.

*Proof:* Fix a neighborhood  $N$  of 0 in  $Z$ . Take sequences  $x_n \rightarrow x_o$  in  $X$  and  $y_n \rightarrow y_o$  in  $Y$ . For each  $x \in X$ , by continuity in  $Y$ ,  $\beta(x, y_n) \rightarrow \beta(x, y_o)$ . Thus, for each  $x \in X$ , the set of values  $\beta(x, y_n)$  is *bounded* in  $Z$ . The linear functionals  $x \rightarrow \beta(x, y_n)$  are *equicontinuous*, by Banach-Steinhaus, so there is a neighborhood  $U$  of 0 in  $X$  so that  $b_n(U) \subset N$  for all  $n$ . In the identity

$$\beta(x_n, y_n) - \beta(x_o, y_o) = \beta(x_n - x_o, y_n) + \beta(x_o, y_n - y_o)$$

we have  $x_n - x_o \in U$  for large  $n$ , and  $\beta(x_n - x_o, y_o) \in N$ . Also, by continuity in  $Y$ ,  $\beta(x_o, y_n - y_o) \in N$  for large  $n$ . Thus,  $\beta(x_n, y_n) - \beta(x_o, y_o) \in N + N$ , proving *sequential* continuity. Since  $X \times Y$  is metrizable, sequential continuity implies continuity. ///



## 5. Appendix: non-existence of tensor products of Hilbert spaces

*Tensor products of infinite-dimensional Hilbert spaces do not exist.*

That is, for infinite-dimensional Hilbert spaces  $V, W$ , **there is no** Hilbert space  $X$  and continuous bilinear map  $j : V \times W \rightarrow X$  such that, for every continuous bilinear  $V \times W \rightarrow Y$  to a Hilbert space  $Y$ , there is a unique continuous linear  $X \rightarrow Y$  fitting into the commutative diagram

$$\begin{array}{ccc} & X & \\ & \uparrow & \searrow \\ j & & \\ & V \times W & \longrightarrow Y \end{array}$$

That is, *there is no tensor product in the category of Hilbert spaces and continuous linear maps.*

Yes, it is possible to put an inner product on the algebraic tensor product  $V \otimes_{\text{alg}} W$ , by

$$\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \cdot \langle w, w' \rangle$$

and extending. The completion  $V \otimes_{\text{HS}} W$ , often denoted  $V \widehat{\otimes} W$ , of  $V \otimes_{\text{alg}} W$  with respect to the associated norm, is a Hilbert space, identifiable with Hilbert-Schmidt operators  $V \rightarrow W^*$ . However, this Hilbert space fails to have the universal property in the categorical characterization of tensor product, as we see below. This Hilbert space  $H$  is important in its own right, but is widely misunderstood as being a tensor product in the categorical sense.

The non-existence of tensor products of infinite-dimensional Hilbert spaces is important *in practice*, not only as a cautionary tale<sup>[6]</sup> about naive category theory, insofar as it leads to Grothendieck's idea of *nuclear spaces*, which *do* admit tensor products.

*Proof:* First, we review the point that the Hilbert-Schmidt tensor product  $H = V \widehat{\otimes} W$  is *not* a Hilbert-space tensor product. For simplicity, suppose that  $V, W$  are *separable*, in the sense of having countable Hilbert-space bases.

Choice of such bases allows an identification of  $W$  with the continuous linear Hilbert space dual  $V^*$  of  $V$ . Then we have the continuous bilinear map  $V \times V^* \rightarrow \mathbb{C}$  by  $v \times \lambda \rightarrow \lambda(v)$ . The algebraic tensor product  $V \otimes_{\text{alg}} W$  injects to  $H = V \widehat{\otimes} V^*$ , and the image is identifiable with the *finite-rank* maps  $V \rightarrow V$ . The linear map  $T : H \rightarrow \mathbb{C}$  induced on the image of  $V \otimes_{\text{alg}} V^*$  is *trace*. If  $H = V \widehat{\otimes} V^*$  were a Hilbert-space tensor product, the trace map would extend continuously to it from finite-rank operators. However, there are many Hilbert-Schmidt operators that are not of trace class. For example, letting  $e_i$  be an orthonormal basis, the element

$$\sum_n \frac{1}{n} \cdot e_n \otimes e_n \in V \widehat{\otimes} V^*$$

does not have a finite trace, since  $\sum_{n \leq N} 1/n \sim \log N$ . In other words, the difficulty is that

$$T\left(\sum_{a \leq n \leq b} \frac{1}{n} \cdot e_n \otimes e_n\right) = \sum_{a \leq n \leq b} \frac{1}{n} \cdot T(e_n \otimes e_n) = \sum_{a \leq n \leq b} \frac{1}{n}$$

[6] Many of us are not accustomed to worry about *existence* of objects defined by universal mapping properties, because we proved their existence by set-theoretic *constructions* of them, long before becoming aware of mapping-property characterizations. Much as naive set theory does not lead to paradoxes without effort, naive category theory's recharacterization of objects close to prior experience rarely describes non-existent objects. Nevertheless, the present example is genuine.

Thus, the partial sums of  $\sum_n \frac{1}{n} e_n \otimes e_n$  form a Cauchy sequence, but the values of  $T$  on the partial sums go to  $+\infty$ . Thus, the Hilbert-Schmidt tensor product cannot be a Hilbert-space tensor product.

Now we show that no *other* Hilbert space can be a tensor product, by comparing to the Hilbert-Schmidt tensor product.

Let  $V \times W \rightarrow X$  be a purported Hilbert-space tensor product, and, again, let  $W$  be the dual of  $V$ , without loss of generality. By assumption, the continuous bilinear injection  $V \times V^* \rightarrow V \otimes_{\text{HS}} V^*$  induces a unique continuous linear map  $T : X \rightarrow V \otimes_{\text{HS}} V^*$  fitting into a commutative diagram

$$\begin{array}{ccc}
 & X & \\
 & \uparrow & \dashrightarrow \\
 & & V \otimes_{\text{alg}} V^* \\
 & \nearrow & \dashrightarrow \\
 V \times V^* & \xrightarrow{\quad} & V \otimes_{\text{HS}} V^*
 \end{array}$$

(The map  $T$  is indicated by a dashed arrow from  $X$  to  $V \otimes_{\text{HS}} V^*$ .)

The linear map  $V \otimes_{\text{alg}} V^* \rightarrow V \otimes_{\text{HS}} V^*$  is injective, since  $V \otimes_{\text{HS}} V^*$  is a completion of  $V \otimes_{\text{alg}} V^*$ . Thus, unsurprisingly,  $V \otimes_{\text{alg}} V^* \rightarrow X$  is necessarily injective. The uniqueness of the linear induced maps implies that the image of  $V \otimes_{\text{alg}} V^*$  is *dense* in  $X$ . Also,  $T : X \rightarrow V \otimes_{\text{HS}} V^*$  is the identity on the copies of  $V \otimes_{\text{alg}} V^*$  imbedded in  $X$  and  $V \otimes_{\text{HS}} V^*$ . Let  $T^* : V \otimes_{\text{HS}} V^* \rightarrow X$  be the adjoint of  $T$ , defined by

$$\langle x, T^* y \rangle_X = \langle Tx, y \rangle_{V \otimes_{\text{HS}} V^*}$$

On the imbedded copies of  $V \otimes_{\text{alg}} V^*$

$$\langle v \otimes \lambda, T^*(w \otimes \mu) \rangle_X = \langle T(v \otimes \lambda), w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*} = \langle v \otimes \lambda, w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*} \quad (\text{for } v, w \in V \text{ and } \lambda, \mu \in V^*)$$

Given  $v \in V$  and  $\lambda \in V^*$ , the orthogonal complement  $(v \otimes \lambda)^\perp$  is the closure of the span of monomials  $v' \otimes \lambda'$  where *either*  $v' \perp v$  or  $\lambda' \perp \lambda$ . For such  $v' \otimes \lambda'$ ,

$$0 = \langle v' \otimes \lambda', v \otimes \lambda \rangle_H = \langle T(v' \otimes \lambda'), v \otimes \lambda \rangle_H = \langle v' \otimes \lambda', T^*(v \otimes \lambda) \rangle_X$$

Thus, for any monomial  $v \otimes \lambda$ , the image  $T^*(v \otimes \lambda)$  is a scalar multiple of  $v \otimes \lambda$ . The same is true of monomials  $(v + w) \otimes (\lambda + \mu)$ . Taking  $v, w$  linearly independent and  $\lambda, \mu$  linearly independent and expanding shows that the scalars do not depend on  $v, \lambda$ . Thus,  $T^*$  is a scalar on  $V \otimes_{\text{alg}} V^*$ .

That is, there is a (necessarily real) constant  $C$  such that

$$C \cdot \langle v \otimes \lambda, w \otimes \mu \rangle_X = \langle v \otimes \lambda, T^*(w \otimes \mu) \rangle_X = \langle T(v \otimes \lambda), w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*} = \langle v \otimes \lambda, w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*}$$

since  $T$  identifies the imbedded copies of  $V \otimes_{\text{alg}} V^*$ . That is, up to the constant  $C$ , the inner products from  $X$  and  $V \otimes_{\text{HS}} V^*$  restrict to the same hermitian form on  $V \otimes_{\text{alg}} V^*$ . Thus, any putative tensor product  $X$  differs from  $V \otimes_{\text{HS}} V^*$  only by scaling. However, we saw that the natural pairing  $V \times V^* \rightarrow \mathbb{C}$  does not factor through a continuous linear map  $V \otimes_{\text{HS}} V^* \rightarrow \mathbb{C}$ , because there exist Hilbert-Schmidt maps not of trace class.

Thus, there is no tensor product of infinite-dimensional Hilbert spaces. ///

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