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01. Review of metric spaces and point-set topology

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1. *Euclidean spaces*

Let $\mathbb{R}^n$ be the usual Euclidean n-space, that is, ordered n-tuples $x = (x_1, \ldots, x_n)$ of real numbers. In addition to vector addition (termwise) and scalar multiplication, we have the usual *distance function* on $\mathbb{R}^n$, in coordinates $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, defined by

$$d(x, y) \;=\; \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

Of course there is visible *symmetry* $d(x, y) = d(y, x)$, and *positivity*: $d(x, y) = 0$ only for $x = y$. The *triangle inequality*

$$d(x, z) \;\le\; d(x, y) + d(y, z)$$

is not trivial to prove. In the one-dimensional case, the triangle inequality is an inequality on absolute values, and can be proven case-by-case. In $\mathbb{R}^n$, it is best to use the following set-up. The usual *inner product* (or *dot-product*) on $\mathbb{R}^n$ is

$$x \cdot y \;=\; \langle x, y\rangle \;=\; \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle \;=\; x_1y_1 + \ldots + x_ny_n$$

and $|x|^2 = \langle x, x\rangle$. *Context* distinguishes the *norm* $|x|$ of $x \in \mathbb{R}^n$ from the usual absolute value $|c|$ on real or complex numbers c. The distance is expressible as

$$d(x, y) \;=\; |x - y|$$

The inner product $\langle x, y\rangle$ is *linear* in both arguments: in the first argument

$$\langle x + x', y\rangle \;=\; \langle x, y\rangle + \langle x', y\rangle \qquad \langle cx, y\rangle \;=\; c \cdot \langle x, y\rangle \qquad (\text{for } x, x', y \in \mathbb{R}^n \text{ and scalar } c)$$

and similarly for the second argument. The triangle inequality will be a corollary of the following universally-useful inequality:

[1.1] Claim: *(Cauchy-Schwarz-Bunyakowsky inequality)* For $x, y \in \mathbb{R}^n$,

$$|\langle x, y\rangle| \;\le\; |x| \cdot |y|$$

Assuming that neither x nor y is 0, *strict* inequality holds unless x and y are scalar multiples of each other.

Proof: If $|y| = 0$, the assertions are trivially true. Thus, take $y \neq 0$. With real t, consider the quadratic polynomial function

$$f(t) \;=\; |x - ty|^2 \;=\; |x|^2 - 2t\langle x, y\rangle + t^2|y|^2$$

Certainly $f(t) \geq 0$ for all $t \in \mathbb{R}$, since $|x - ty| \geq 0$. Its minimum occurs where $f'(t) = 0$, namely, where $-2\langle x, y\rangle + 2t|y|^2 = 0$. This is where $t = \langle x, y\rangle/|y|^2$. Thus,

$$0 \;\leq\; (\text{minimum}) \;\leq\; f(\langle x,y\rangle/|y|^2) \;=\; |x|^2 - 2\frac{\langle x,y\rangle}{|y|^2}\langle x,y\rangle + \Big(\frac{\langle x,y\rangle}{|y|^2}\Big)^2 \cdot |y|^2 \;=\; |x|^2 - \Big(\frac{\langle x,y\rangle}{|y|^2}\Big)^2 \cdot |y|^2$$

Multiplying out by $|y|^2$,

$$0 \;\leq\; |x|^2 \cdot |y|^2 - \langle x,y\rangle^2$$

which gives the inequality. Further, for the inequality to be an *equality*, it must be that $|x - ty| = 0$, so x is a multiple of y. ///

[1.2] Remark: We did not use properties of $\mathbb{R}^n$, only of the inner product!

[1.3] Corollary: *(Triangle inequality)* For $x, y, z \in \mathbb{R}^n$,

$$|x + y| \;\leq\; |x| + |y|$$

Therefore,

$$d(x,z) \;=\; |x - z| \;=\; |(x-y) - (z-y)| \;\leq\; |x-y| + |z-y| \;=\; d(x,y) + d(y,z)$$

Proof: With the Cauchy-Schwarz-Bunyakowsky inequality in hand, this is a direct computation:

$$|x+y|^2 \;=\; \langle x+y, x+y\rangle \;=\; \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \;=\; |x|^2 + 2\langle x,y\rangle + |y|^2 \;\leq\; |x|^2 + 2|\langle x,y|\rangle + |y|^2$$

$$\leq\; |x|^2 + 2|x|\cdot|y| + |y|^2 \;=\; (|x| + |y|)^2$$

Taking positive square roots gives the result. ///

The *open ball* B of radius $r > 0$ centered at a point y is

$$B \;=\; \{x \in \mathbb{R}^n \;:\; d(x,y) < r\}$$

The *closed ball* $\overline{B}$ of radius $r > 0$ centered at a point y is

$$\overline{B} \;=\; \{x \in \mathbb{R}^n \;:\; d(x,y) \leq r\}$$

Obviously in many regards the two are barely different from each other. However, the fact that the *closed* ball includes its *boundary* (in both an intuitive an technical sense as below) the sphere

$$S^{n-1} \;=\; \{x \in \mathbb{R}^n \;:\; d(x,y) = r\}$$

while the *open* ball does *not*. A different distinction is what we'll exploit most directly:

[1.4] Corollary: For any point x in an open ball B in $\mathbb{R}^n$, for sufficiently small radius $\varepsilon > 0$ the open ball of radius ε centered at x is contained in B.

Proof: This is essentially the triangle inequality. Let B be the open ball of radius r centered at y. Then $x \in B$ if and only if $|x - y| < r$. Thus, we can take $\varepsilon > 0$ such that $|x - y| + \varepsilon < r$. For $|z - x| < \varepsilon$, by the triangle inequality

$$|z-y| \;\leq\; |z-x| + |x-y| \;<\; \varepsilon + |x-y| \;<\; r$$

That is, the open ball of radius ε at x is inside B. ///

An *open set* in $\mathbb{R}^n$ is any set with the property observed in the latter corollary, namely a set U in $\mathbb{R}^n$ is *open* if for every x in U there is an open ball centered at x contained in U.

This definition allows us to rewrite the epsilon-delta definition of *continuity* in a useful form:

[1.5] Claim: A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is *continuous* if and only if the inverse image

$$f^{-1}(U) = \{x \in \mathbb{R}^m \,:\, f(x) \in U\}$$

of every *open* set U in $\mathbb{R}^n$ is *open* in $\mathbb{R}^m$. *(We prove this below for general metric spaces.)* ///

Some properties of open sets in $\mathbb{R}^n$ that will be abstracted:

[1.6] Claim: The union of an *arbitrary* set of open subsets of $\mathbb{R}^n$ is open. The intersection of a *finite* set of open subsets of $\mathbb{R}^n$ is open.

Proof: A point $x \in \mathbb{R}^n$ is in the union U of an arbitrary set $\{U_\alpha : \alpha \in A\}$ of open subsets of $\mathbb{R}^n$ exactly when there is some U_α so that $x \in U_\alpha$. Then a small-enough open ball B centered at x is inside U_α, so $B \subset U_\alpha \subset U$.

For x in the intersection $I = U_1 \cap \ldots \cap U_m$ of a finite number of opens, let $\varepsilon_j > 0$ such that the open ε_j-ball at x is contained in U_j. Let ε be the minimum of the ε_j. The minimum of a *finite* set of (strictly) positive real numbers is still (strictly) positive, so $\varepsilon > 0$, and the ε-ball at x is contained inside every ε_j-ball at x, so is contained in the intersection. ///

One of many equivalent ways to say that a set E in $\mathbb{R}^n$ is *bounded* is that it is contained in some (sufficiently large) *ball.* [1] At various technical points in advanced calculus, we find ourselves caring about *closed and bounded* sets, and perhaps proving the *Heine-Borel property* or *Bolzano-Weierstraß property*[2]

[1.7] Theorem: A set E in $\mathbb{R}^n$ is closed and bounded *if and only if* every sequence of points in E has a *convergent subsequence.* ///

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2. *Metric spaces*

By design, the previous discussion of Euclidean spaces made minimal use of particular features of Euclidean space. This allows *abstraction* of some relevant features in a manner that uses our intuition about Euclidean spaces to suggest things about less familiar spaces. The process of abstraction has several different stopping places, and this section looks at one of the first.

We can abstract the *distance function* on $\mathbb{R}^n$ usefully, as follows. For a set X be a set, a non-negative-real-valued function

$$d : X \times X \longrightarrow \mathbb{R}$$

is a *distance function* if it satisfies the conditions

$$\begin{cases} d(x,y) \geq 0 & \text{(with equality only for } x = y) \quad \textit{(positivity)} \\ d(x,y) = d(y,x) & \textit{(symmetry)} \\ d(x,z) \leq d(x,y) + d(y,z) & \textit{(triangle inequality)} \end{cases}$$

[1] A few moments' thought show that it does not matter where the ball is centered, nor whether the ball is closed or open.

[2] This property is not at all trivial to prove, especially from an elementary viewpoint.

for all points $x, y, z \in X$. Such a distance function is also called a *metric*. The set X with the metric d is a *metric space*.

In analogy with the situation for $\mathbb{R}$ and $\mathbb{R}^n$, a sequence $\{x_n\}$ in a metric space X is *convergent* to $x \in X$ when, for every $\varepsilon > 0$, there is n_o such that, for all $n \geq n_o$, $|x_n - x| < \varepsilon$. Likewise, a sequence $\{x_n\}$ in X is a *Cauchy* sequence when, for all $\varepsilon > 0$, there is n_o such that for all $m, n \geq n_o$, $|x_m - x_n| < \varepsilon$. A metric space is *complete* if every Cauchy sequence is convergent.

The following standard lemma makes a bit of intuition explicit:

[2.1] Lemma: Let $\{x_i\}$ be a Cauchy sequence in a metric space X, d converging to x in X. Given $\varepsilon > 0$, let N be sufficiently large such $d(x_i, x_j) < \varepsilon$ for $i, j \geq N$. Then $d(x_i, x) \leq \varepsilon$ for $i \geq N$.

Proof: Let $\delta > 0$ and take $j \geq N$ also large enough such that $d(x_j, x) < \delta$. Then for $i \geq N$ by the triangle inequality

$$d(x_i, x) \ \leq \ d(x_i, x_j) + d(x_j, x) \ < \ \varepsilon + \delta$$

Since this holds for every $\delta > 0$ we have the result. ///

[2.2] Example: Variants of the usual Euclidean metric on $\mathbb{R}^n$ also make sense:

$$d_1(x, y) \ = \ |x_1 - y_1| + \ldots + |x_n - y_n| \qquad\qquad d_\infty(x, y) \ = \ \max_i |x_i - y_i|$$

In fact, the triangle inequality for these metrics are easy to prove, needing just the triangle inequality for the absolute value on $\mathbb{R}$. Later, we will see [3] that

$$d_p(x, y) \ = \ \Big(|x_1 - y_1|^p + \ldots + |x_n - y_n|^p\Big)^{1/p} \qquad \text{(for } 1 \leq p < \infty)$$

also gives a metric.

[2.3] Example: A *discrete set* or *discrete metric space* X is one in which (roughly) no two distinct points are close to each other. That is, for each $x \in X$ there should be a bound $\delta_x > 0$ such that $d(x, y) \geq \delta_x$ for all $y \neq x$ in X. For example, the set $\mathbb{Z}$ of integers, with the natural distance

$$d(x, y) \ = \ |x - y| \qquad \text{(with usual absolute value)}$$

has the property that $|x - y| \geq 1$ for distinct integers. Every discrete metric space is complete.

[2.4] Example: *Any* set X can be made into a *discrete* metric space by defining

$$d(x, y) \ = \ \begin{cases} 1 & (\text{for } x \neq y) \\ 0 & (\text{for } x = y) \end{cases}$$

This is obviously positive and symmetric, and satisfies the triangle inequality condition for silly reasons. Little is learned from this example except that it is possible to do such things.

[2.5] Example: The collection $C^o[a, b]$ of continuous functions [4]) on an interval $[a, b]$ on the real line can be given the metric

$$d(f, g) \ = \ \sup_{x \in [a,b]} |f(x) - g(x)|$$

[3] The triangle inequality for such metrics is an instance of the *Hölder inequality*.

[4] Throughout discussion of these examples, it doesn't matter much whether we think of real-valued functions or complex-valued functions.

Positivity and symmetry are easy, and the triangle inequality is not hard, either. This metric space is *complete*, because a Cauchy sequence is a *uniformly pointwise convergent* sequence of continuous functions.

[2.6] Example: The collection $C^o(\mathbb{R})$ of continuous functions [5] on the *whole* real line does *not* have an obvious candidate for a metric, since the *sup* metric of the previous example may give infinite values. Yet there is the metric

$$d(f,g) \;=\; \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{|x|\le n} |f(x)-g(x)|}{1+\sup_{|x|\le n} |f(x)-g(x)|}$$

This metric space is complete, for similar reasons as $C^o[a,b]$.

[2.7] Example: A sort of infinite-dimensional analogue of the standard metric on $\mathbb{R}^n$ is the space ℓ^2, the collection of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ of complex numbers such that $\sum_{n\ge 1} |\alpha_n|^2 < +\infty$. The metric is

$$d(\alpha,\beta) \;=\; \sqrt{\sum_{n\ge 1} |\alpha_n - \beta_n|^2}$$

In fact, ℓ^2 is a vector space, being closed under addition and under scalar multiplication, with inner product

$$\langle \alpha, \beta\rangle \;=\; \sum_{n\ge 1} \alpha_n \cdot \overline{\beta}_n$$

The associated *norm* is $|\alpha| = \langle \alpha,\alpha\rangle^{\frac{1}{2}}$, and $d(\alpha,\beta) = |\alpha-\beta|$. The Cauchy-Schwarz-Bunyakowsky holds for ℓ^2, by the same proof as given earlier, and proves the triangle inequality. This metric space is complete.

[2.8] Example: For $1 \le p < \infty$, the sequence spaces ℓ^p is

$$\ell^p \;=\; \{x = (x_1, x_2, \ldots) \;:\; \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

with metric

$$d_p(x,y) \;=\; \Big(\sum_{i=1}^{\infty} |x_i - y_i|^p\Big)^{1/p}$$

Proof of the triangle inequality needs *Hölder's inequality.* These metric spaces are complete. Unlike the case of varying metrics on $\mathbb{R}^n$, the underlying sets ℓ^p are not the same. For example, ℓ^2 is strictly larger than ℓ^1.

[2.9] Example: Even before having a modern notion of measure and integral, a partial analogue of ℓ^2 can be formulated: on $C^o[a,b]$, form an inner product

$$\langle f, g\rangle \;=\; \int_a^b f(x)\, \overline{g(x)}\, dx$$

It is easy to check that this does give a hermitian inner product. The L^2 norm is $|f|_{L^2} = \langle f,f\rangle^{\frac{1}{2}}$, and the distance function is $d(f,g) = |f-g|$. The basic properties of a metric are immediate, except that the triangle inequality needs the integral form of the Cauchy-Schwarz-Bunyakowsky inequality, whose proof is the same as that given earlier. This metric space is *not* complete, because there are sequences of continuous functions that are Cauchy in this L^2 metric (but not in the $C^o[a,b]$ metric) and do not converge to a continuous

[5] Real-valued or complex-valued, for example.

function. For example, we can make piecewise-linear continuous functions approaching the discontinuous function that is 0 on $[a, \frac{a+b}{2}]$ and 1 on $[\frac{a+b}{2}, b]$, by

$$f_n(x) \;=\; \begin{cases} 0 & (\text{for } a \le x \le \frac{a+b}{2} - \frac{1}{n}) \\ \frac{n}{2} \cdot \left(x - \left(\frac{a+b}{2} - \frac{1}{n}\right)\right) & (\text{for } \frac{a+b}{2} - \frac{1}{n} \le x \le \frac{a+b}{2} + \frac{1}{n}) \\ 1 & (\text{for } \frac{a+b}{2} + \frac{1}{n} \le x \le b) \end{cases}$$

(Draw a picture.) The pointwise limit is 0 to the left of the midpoint, and 1 to the right. Despite the fact that the pointwise limit does not exist at the midpoint,

$$d_2(f_i, f_j)^2 \;\le\; \int_{\frac{a+b}{2}-\frac{1}{n}}^{\frac{a+b}{2}+\frac{1}{n}} 1\; dx \;\le\; \frac{2}{n} \qquad\qquad (\text{for } i, j \ge n)$$

which goes to 0 as $n \to \infty$. That is, $\{f_n\}$ is Cauchy in the L^2 metric, but does not converge to a continuous function.

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3. *Completions of metric spaces*

Again, a metric space X, d is *complete* when every Cauchy sequence is convergent. Completeness is a convenient feature, because then we can take limits without leaving the space. As in the example of $C^o[a, b]$ with the L^2 inner product, we might want to imbed a non-complete metric space in a complete one in an optimal and universal way.

A traditional notion of the *completion* of a metric space X is a *construction* of a complete metric space $\widetilde{X}$ with a distance-preserving injection $j : X \to \widetilde{X}$ so that $j(X)$ is *dense* in $\widetilde{X}$, in the sense that every point of $\widetilde{X}$ is the limit of a Cauchy sequence in $j(X)$.

The intention is that every Cauchy sequence has a limit, so we should (somehow!) *adjoin* points as needed for these limits. However, different Cauchy sequences may happen to have the same limit.

Thus, we want an equivalence relation on Cauchy sequences that says they should have the same limit, even without knowing the limit exists or having somehow constructed or adjoined the limit point.

Define an equivalence relation $\sim$ on the set C of Cauchy sequences in X, by

$$\{x_s\} \sim \{y_t\} \iff \lim_s d(x_s, y_s) \;=\; 0$$

Attempt to define a metric on the set $C/\!\sim$ of equivalence classes by

$$d(\{x_s\}, \{y_t\}) \;=\; \lim_s d(x_s, y_s)$$

We must verify that this is well-defined on the quotient $C/\!\sim$ and gives a metric. We have an injection $j : X \to C/\!\sim$ by

$$x \to \{x, x, x, \ldots\} \bmod \sim$$

[3.1] Claim: $j : X \to C/\!\sim$ is a completion of X.

Proof: Grant for the moment that the distance function on $\widetilde{X} = C/\!\sim$ is well-defined, and is complete, and show that it has the property of a completion of X. To this end, let $f : X \to Y$ be a *uniformly* continuous map to a complete metric space Y.

Given $z \in \widetilde{X}$, choose a Cauchy sequence x_k in X with $j(x_k)$ converging to z, and *try* to define $F : \widetilde{X} \to Y$ in the natural way, by

$$F(z) \;=\; \lim_k f(x_k)$$

Since f is *uniformly* continuous, $f(x_k)$ is Cauchy in Y, and by completeness of Y has a limit, so $F(z)$ *exists*, at least if well-defined.

For well-definedness of $F(z)$, for x_k and x'_k two Cauchy sequences whose images $j(x_k)$ and $j(x'_k)$ approach z, since j is an isometry eventually x_k is close to x'_k, so $f(x_k)$ is eventually close to $f(x'_k)$ in Y, showing $F(z)$ is well-defined.

We saw that every element of $\widetilde{X}$ is a limit of a Cauchy sequence $j(x_k)$ for x_k in X, and *any* continuous $\widetilde{X} \to Y$ respects limits, so F is the only possible extension of f to $\widetilde{X}$.

The obvious argument will show that F is continuous. Namely, let $z, z' \in \widetilde{X}$, with Cauchy sequences x_t and x'_t approaching z and z'. Given $\varepsilon > 0$, by uniform continuity of F, there is N large enough such that $d_Y(F(j(x_r)), F(j(x_s))) < \varepsilon$ and $d_Y(F(j(x'_r)), F(j(x'_s))) < \varepsilon$ for $r, s \geq N$. From the lemma above (!), for such r even in the limit the strict inequalities are at worst non-strict inequalities:

$$d_Y(f(x_r), F(z)) \;\leq\; \varepsilon \qquad \text{and} \qquad d_Y(f(x'_r), F(z')) \;\leq\; \varepsilon$$

By the triangle inequality, since $f : X \to Y$ is continuous, we can increase r to have $d_X(x_r, x'_r)$ small enough so that $d_Y(f(x_r), f(x'_r)) < \varepsilon$, and then

$$d_Y(F(z), F(z')) \;\leq\; d_Y(F(z), f(x_r)) + d_Y(f(x_r), f(x'_r)) + d_Y(f(x'_r), F(z')) \;\leq\; \varepsilon + \varepsilon + \varepsilon$$

Since $j : X \to \widetilde{X}$ is an isometry,

$$d_X(x_r, x'_r) \;=\; d_{\widetilde{X}}(j(x_r), j(x'_r)) \;\leq\; d_{\widetilde{X}}(j(x_r), z) + d_{\widetilde{X}}(z, z') + d_{\widetilde{X}}(j(x'_r), z')$$

so

$$d_X(x_r, x'_r) \;\leq\; d_{\widetilde{X}}(z, z') \;+\; 2\varepsilon$$

Thus,

$$d_Y(F(z), F(z')) \;\leq\; d_{\widetilde{X}}(z, z') + 4\varepsilon \qquad \text{(for all } \varepsilon > 0)$$

Thus, F is continuous. Granting that $\widetilde{X} = C/\!\sim$ is complete, etc., it is a completion of X.

It remains to prove that the apparent metric on $\widetilde{X}$ truly is a metric, and that $\widetilde{X}$ is complete.

First, the limit in attempted definition

$$d(\{x_s\}, \{y_t\}) \;=\; \lim_s d(x_s, y_s)$$

does exist: given $\varepsilon > 0$, take N large enough so that $d(x_i, x_j) < \varepsilon$ and $d(y_i, y_j) < \varepsilon$ for $i, j \geq N$. By the triangle inequality,

$$d(x_i, y_i) \;\leq\; d(x_i, x_N) + d(x_N, y_N) + d(y_N, y_i) \;<\; \varepsilon + d(x_N, y_N) + \varepsilon$$

Similarly,

$$d(x_i, y_i) \;\geq\; -d(x_i, x_N) + d(x_N, y_N) - d(y_N, y_i) \;>\; -\varepsilon + d(x_N, y_N) - \varepsilon$$

Thus, unsurprisingly,

$$\Big| d(x_i, y_i) - d(x_N, y_N) \Big| \;<\; 2\varepsilon$$

and the sequence of real numbers $d(x_i, y_i)$ is Cauchy, so convergent.

Similarly, when $\lim_i d(x_i, y_i) = 0$, then $\lim_i d(x_i, z_i) = \lim_i d(y_i, z_i)$ for any other Cauchy sequence z_i, so the distance function is *well-defined* on $C/\sim$.

The positivity and symmetry for the alleged metric on $C/\sim$ are immediate. For triangle inequality, given x_i, y_i, z_i and $\varepsilon > 0$, let N be large enough so that $d(x_i, x_j) < \varepsilon$, $d(y_i, y_j) < \varepsilon$, and $d(z_i, z_j) < \varepsilon$ for $i, j \geq N$. As just above,

$$\Big| d(\{x_s\}, \{y_s\}) - d(x_i, y_i) \Big| \;<\; 2\varepsilon$$

Thus,

$$d(\{x_s\}, \{y_s\}) \;\leq\; 2\varepsilon + d(x_N, y_N) \;\leq\; 2\varepsilon + d(x_N, z_N) + d(z_N, y_N) \;\leq\; 2\varepsilon + d(\{x_s\}, \{z_s\}) + 2\varepsilon + d(\{z_s\}, \{y_s\}) + 2\varepsilon$$

This holds for all $\varepsilon > 0$, so we have the triangle inequality.

Finally, perhaps anticlimactically, the completeness. Given Cauchy sequences $c_s = \{x_{sj}\}$ in X such that $\{c_s\}$ is Cauchy in $C/\sim$, for each s we will choose large-enough $j(s)$ such that the diagonal-ish sequence $y_\ell = x_{\ell, j(\ell)}$ is a Cauchy sequence in X to which $\{c_s\}$ converges.

Given $\varepsilon > 0$, take i large enough so that $d(c_s, c_t) < \varepsilon$ for all $s, t \geq i$. For each i, choose $j(i)$ large enough so that $d(x_{ij}, x_{ij'}) < \varepsilon$ for all $j, j' \geq j(i)$. Let $c = \{x_{i,j(i)} : i = 1, 2, \ldots\}$. For $s \geq i$,

$$d(c_s, c) \;=\; \lim_\ell d(x_{s\ell}, x_{\ell, j(\ell)}) \;\leq\; \sup_{\ell \geq i} d(x_{s\ell}, x_{\ell, j(\ell)}) \;\leq\; \sup_{\ell \geq i} \Big(d(x_{s\ell}, x_{s, j(\ell)}) + d(x_{s, j(\ell)}, x_{\ell, j(\ell)}) \Big) \;\leq\; 2\varepsilon$$

This holds for all $\varepsilon > 0$, so $\lim_s c_s = c$, and $C/\sim$ is complete. ///

Many natural metric spaces are complete without any need to complete them. The historically notable exception was $\mathbb{Q}$ itself, completed to $\mathbb{R}$. A slightly more recent example:

[3.2] Example: One description of the space $L^2[a, b]$ is as the completion of $C^o[a, b]$ with respect to the L^2 norm above. The more common description depends on notions of *measurable function* and *Lebesgue integral*, and presents the space as equivalence classes of functions, having somewhat ambiguous pointwise values.

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4. *Topologies of metric spaces*

The notion of metric space allows a useful generalization of the notion of *continuous function* via the obvious analogue of the epsilon-delta definition:

A function or map $f : X \to Y$ from one metric space (X, d_X) to another metric space (Y, d_Y) is *continuous* at a point $x_o \in X$ when, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_X(x, x_o) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(x_o)) < \varepsilon$$

In a metric space (X, d), the *open ball* of radius $r > 0$ centered at a point y is

$$\{x \in X \,:\, d(x, y) < r\}$$

The *closed ball* of radius $r > 0$ centered at a point y is

$$\{x \in X \,:\, d(x, y) \leq r\}$$

As in $\mathbb{R}^n$, in many regards the two are barely different from each other. However, the *closed* ball includes the *sphere*

$$\{x \in X \,:\, d(x, y) = r\}$$

while the *open* ball does *not*. A different distinction is what we'll exploit most directly:

[4.1] Claim: For any point x in an open ball B in X, for sufficiently small radius $\varepsilon > 0$ the open ball of radius ε centered at x is contained in B. *(As for $\mathbb{R}^n$, this follows immediately by use of the triangle inequality.* ///

An *open set* in X is any set with the property observed in this proposition. That is, a set U in X is *open* if for every x in U there is an open ball centered at x contained in U.

This definition allows us to rewrite the epsilon-delta definition of *continuity* in a form that will apply in more general topological spaces:

[4.2] Claim: A function $f : X \to Y$ from one metric space to another is *continuous* in the ε-δ sense if and only if the inverse image

$$f^{-1}(U) \;=\; \{x \in \mathbb{R}^m \,:\, f(x) \in U\}$$

of every *open* set U in Y is *open* in X.

Proof: On one hand, suppose f is continuous in the ε-δ sense. For U open in Y and $x \in f^{-1}(U)$, with $f(x) = y$, let $\varepsilon > 0$ be small enough so that the ε-ball at y is inside U. Take $\delta > 0$ small enough so that, by the ε-δ definition of continuity, the δ-ball B at x has image $f(B)$ inside the ε-ball at y. Then $x \in B \subset f^{-1}(U)$. This holds for every $x \in f^{-1}(U)$, so $f^{-1}(U)$ is open.

On the other hand, suppose $f^{-1}(U)$ is open for every open $U \subset Y$. Given $x \in X$ and $\varepsilon > 0$, let U be the ε-ball at $f(x)$. Since $f^{-1}(U)$ is open, there is an open ball B at x contained in $f^{-1}(U)$. Let $\delta > 0$ be the radius of B. ///

A set E in a metric space X is *closed* if and only its *complement*

$$E^c \;=\; X - E \;=\; \{x \in X \,:\, x \notin E\}$$

is *open*.

A set E in a metric space X is *bounded* when it is contained in some (sufficiently large) *ball*. This makes sense in general metric spaces, but does not have the same implications.

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5. *General topological spaces*

Many of the ideas and bits of terminology for metric spaces make sense and usefully extend to more general situations. Some do not.

[5.1] A topology on a set X is a collection τ of subsets of X, called the *open sets*, such that X itself and the empty set ϕ are in τ, *arbitrary unions* of elements of τ are in τ, and *finite intersections* of elements of τ are in τ. A set X with an explicitly or implicitly specified topology is a *topological space*.

[5.2] A continuous map $f : X \to Y$ for topological spaces X, Y is a set-map so that inverse images $f^{-1}(U)$ of opens U in Y are open in X.

Uniform continuity of functions or maps has no natural formulation in general topological spaces, in effect because we have no device by which to compare the topology at varying points, unlike the case of metric spaces, where there is a common notion of distance that does allow such comparisons.

[5.3] Closed sets in a topological space are exactly the complements of open sets. Arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed.

[5.4] A basis for a topology is a collection of (open) subsets so that any open set is a union of the (open) sets in the basis. In a metric space, the open balls of all possible sizes, at all points, are a natural basis.

[5.5] A neighborhood of a point is any set containing an open set containing the point. Often, one considers only *open* neighborhoods, to avoid irrelevant misunderstandings.

[5.6] A local basis at a point x in a space X is a collection of open neighborhoods of x such that every neighborhood of x *contains* a neighborhood from the collection. In a metric space, the collection of open balls at a given point with *rational radius* is a countable local basis at that point.

[5.7] The closure of a set E (in a topological space X), sometimes denoted $\overline{E}$, is the intersection of all closed sets containing E. It is a closed set. Equivalently, it is the set of $x \in X$ such that every neighborhood of x meets[6] E. The closure of E contains E.

[5.8] The interior of a set E (in a topological space X) is the union of all open sets contained in it. It is open. Equivalently, it is the set of $x \in X$ such that there is a neighborhood of x inside E. The interior of E is a subset of E.

[5.9] The boundary of a set E (in a topological space X), often denoted ∂E, is the intersection of the closure of E and the closure of the complement of E. Equivalently, it is the set of $x \in X$ such that every neighborhood of x meets both E and the complement of E.

[5.10] A Hausdorff topology is one in which any two points x, y have neighborhoods $U \ni x$ and $V \ni y$ which are disjoint: $U \cap V = \phi$. This is a reasonable condition to impose on a space on which functions should live.

[5.11] Claim: Metric spaces are Hausdorff.

Proof: Given $x \neq y$ in a metric space, let B_1 be the open ball of radius $d(x, y)/2$, and let B_2 the open ball of radius $d(x, y)/2$ at y. For any $z \in B_1 \cap B_2$, by the triangle inequality,

$$d(x,y) \;\le\; d(x,z) + d(z,y) \;<\; \frac{d(x,y)}{2} + \frac{d(x,y)}{2} \;=\; d(x,y)$$

which is impossible. Thus, there is no z in the intersection of these two open neighborhoods of x and y. ///

[5.12] Claim: In Hausdorff spaces, singleton sets $\{x\}$ are closed.

Proof: Fixing x, for $y \neq x$ let U_y be an open neighborhood of y not containing x. (We do not use the open neighborhood of x not meeting U_y.) Then $E = \bigcup_{y \neq x} U_y$ is open, does not contain x, and contains every other point in the space. Thus, E is the complement of the singleton set $\{x\}$ and is open, so $\{x\}$ is closed. ///

[5.13] Convergence of sequences: In a topological space X, a sequence $x_1, x_2, \ldots$ converges to $x_\infty \in X$, written $\lim_n x_n = x_\infty$, if, for every neighborhood U of x_∞, there is an index m such that for all $n \geq m$, $x_n \in U$.

In more general, non-Hausdorff spaces, it is easily possible to have a sequence converge to more than one point, which is fairly contrary to our intention for the notion of *convergence*.

---

[6] A set X *meets* another set Y if $X \cap Y \neq \phi$.

In a *metric* space, the notion of *Cauchy* sequence has a sense, and in a *complete* metric space, the notions of Cauchy sequence and convergent sequence are identical, and there is a unique limit to which such a sequence converges.

In more general, non-Hausdorff spaces, and not-locally-countably-based spaces, things can go haywire in several different ways, which are mostly irrelevant to the situations we care about. Still, one should be aware that not all spaces are Haudorff, and may fail to be countably locally based.

[5.14] Sequentially compact sets E in a topological space X are those such that every sequence has a convergent subsequence (with limit in E).

Although the definition of *convergent* does not directly mention potential difficulties and ambiguities, there are indeed problems in non-Haudorff spaces, and in spaces that fail to have countable local bases.

[5.15] Accumulation points of a subset E of a topological space X are points $x \in X$ such that every neighborhood of x contains infinitely-many elements of E. Every accumulation point of E lies in the *closure* of E, but not vice-versa.

[5.16] Claim: A closed set E is sequentially compact if and only if every sequence in E either has an accumulation point in E, or contains only finitely-many distinct points.

Proof: First, the technicality: if a sequence contains only finitely-many distinct points, it cannot have any accumulation points, but certainly contains convergent subsequences. For a sequence $x_1, x_2, \ldots$ including infinitely-many distinct points, drop any repeated points, so that $x_i \neq x_j$ for all $i \neq j$. For E sequentially compact, there is a subsequence with limit x_∞ in E. Relabel if necessary so that the subsequence is still denoted $x_1, x_2, \ldots$. The subsequence still consists of mutually distinct points. Since $\lim_n x_n = x_\infty$, given a neighborhood U of x_∞, there is m such that $x_n \in U$ for all $n \geq m$. Since $x_m, x_{m+1}, \ldots$ is an infinite set of distinct points, x_∞ is an accumulation point of the subsequence, hence, of the original sequence.

Conversely, if a sequence has an accumulation point, it has a subsequence converging to that accumulation point. ///

[5.17] Compact sets in topological spaces are subsets such that *every open cover has a finite subcover*. That is, K is compact when, for any collection of open sets $\{U_\alpha : \alpha \in A\}$ such that $K \subset \bigcup_{\alpha \in A} U_\alpha$, there is a finite collection $U_{\alpha_1}, \ldots, U_{\alpha_n}$ such that $K \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.

[5.18] Claim: For $f : X \to Y$ continuous and K compact in X, the image $f(K)$ is compact in Y.

Proof: Given an open cover $\{U_\alpha : \alpha \in A\}$ of $f(K)$, the inverse images $f^{-1}(U_\alpha)$ give an open cover of K. Thus, there is a finite subcover $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$. Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a (finite) cover of $f(K)$. ///

Since singleton sets $\{x\}$ are certainly compact, the following generalizes the earlier claim about closedness of singleton sets in Hausdorff spaces:

[5.19] Claim: In Hausdorff spaces, compact sets are *closed*.

Proof: Let E be a compact subset of X. For $y \notin E$, for each $x \in E$, let $U_x \ni y$ be open and $V_x \ni x$ open so that $U_x \cap V_x = \phi$. Then $\{V_x : x \in E\}$ is an open cover of E, with finite subcover $E \subset V_{x_1} \cup \ldots \cup V_{x_n}$. The finite intersection $W_y = U_{x_1} \cap \ldots \cap U_{x_n}$ is open, and disjoint from $V_{x_1} \cup \ldots \cup V_{x_n}$, so is disjoint from E. Thus, W_y is open and contains y. The union $W = \bigcup_{y \notin E} W_y$ is open, and contains every $y \notin E$. Thus E is the complement of an open set, so is closed. ///

[5.20] Claim: In Hausdorff spaces, a *nested* collection of compact sets has non-empty intersection.

Proof: Let X be the ambient space, and K_α the compacts, with index set A *totally ordered*, in the sense A has an order relation $<$ such that for every distinct $\alpha, \beta \in A$, either $\alpha < \beta$ or $\beta < \alpha$. The *nested* condition is that if $\alpha < \beta$ then $K_\alpha \supset K_\beta$. (It can equally well be the opposite direction of containment.) We claim that $\bigcap_\alpha K_\alpha$ is compact.

From above, each K_α is closed, so the complements $U_\alpha = X - K_\alpha$ are open. If $\bigcap_\alpha K_\alpha = \phi$, then $\bigcup_\alpha U_\alpha = X$. In particular, $\bigcup_\alpha U_\alpha \supset K_\beta$ for all indices β. For fixed index α_o, let $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$ be a finite subcover of K_{α_o}, so certainly a cover of $K_{\alpha'}$ for all $\alpha' > \alpha$. Because of the nested-ness, for $\beta = \max\{\alpha_1, \ldots, \alpha_n\}$, $U_\beta = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. But U_β is the complement of K_β, so certainly cannot cover it, contradiction. ///

[5.21] A locally compact topology is one in which every point has a neighborhood with compact closure. This is a reasonable condition to impose on a space on which functions will live. $\mathbb{R}^n$ is locally compact, but the metric space ℓ^2 is *not*. Later, we will see that *no* infinite-dimensional Hilbert space or Banach space is locally compact. That is, *natural spaces of functions* are not usually locally compact, but the physical spaces on which the functions live usually *are* locally compact.

[5.22] Separable topological spaces are those with countable dense subsets. For example, the countable set $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Nearly all topological spaces arising in practice are separable, but most basic results do not directly use this property.

[5.23] Countably-based topological spaces are those with a countable basis. Sometimes such spaces are called *second-countable*. Perhaps counter-intuitively, *first-countable* spaces are those in which every point has a countable *local* basis. Many topological spaces arising in practice are countably-based, but most basic results do not directly use this property.

[5.24] Claim: Separable metric spaces are countably-based. Specifically, for countable dense subset S of metric space X, open balls of *rational* radius centered at points of S form a basis.

Proof: Since there are only countably-many $s \in S$ and only countably many rational radiuses, the set of such open balls is indeed countable.

Fix an open $U \subset X$. Given $x \in U$, let $r > 0$ be sufficiently small so that the open ball at x of radius r is inside U. Let $s_x \in S$ be such that $d(x, s) < r/2$. By density of rational numbers in $\mathbb{R}$, there is a rational number q_x such that $d(x, s) < q_x < r/2$. Thus, by the triangle inequality, the ball B_x at s_x of radius q_x contains x and lies inside the open ball at x of radius r, so $B_x \subset U$.

The union of all B_x over $x \in U$ is a subset of U containing all $x \in U$, so is U itself. ///

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6. *Compactness versus sequential compactness*

In general topological spaces, *compactness* is a stronger condition than *sequential compactness*. First, without any further hypotheses on the spaces, however noting the point that sequential compactness easily fails to be what we anticipate in topological spaces that are not necessarily Hausdorff or locally countably-based:

[6.1] Claim: Compact sets are sequentially compact.

Proof: Given a sequence, if some $y \in E$ is an accumulation point, then there is a subsequence converging to y, and we are done. If no $y \in E$ is an accumulation point of the given sequence, then each $y \in E$ has an open neighborhood U_y such that U_y meets the sequence in only finitely-many points. The sets U_y cover E. For E compact, there is a finite subcover $U_{y_1}, \ldots, U_{y_n}$. Each U_{y_i} contains only finitely-many points of the sequence, so the sequence contains only finitely-many distinct points, so certainly has a convergent subsequence. ///

[6.2] Claim: In a countably-based topological space X, sequentially compact sets are compact.

Proof: Let $E \subset X$ be sequentially compact. The opens in an arbitrary cover of E are (necessarily countable) unions of some of the countably-many opens in the countable basis for X. Thus, it suffices to show that a *countable* cover $E \subset U_1 \cup U_2 \cup \ldots$ admits a finite subcover.

If *no* finite collection of the U_n covers E, then for each $n = 1, 2, \ldots$ there is $e_n \in E$ such that $e_n \notin U_1 \cup \ldots \cup U_n$. Since every e_n does lie in *some* U_i, we can replace $\{e_n\}$ by a subsequence so that $e_i \neq e_j$ for all $i \neq j$, and still $e_n \notin U_1 \cup \ldots \cup U_n$.

By sequential compactness, $e_1, e_2, \ldots$ has a convergent subsequence, with limit $e_\infty \in E$. The point e_∞ lies in some U_m. Thus, there would be infinitely-many indices n such that $e_n \in U_m$. This is impossible, since $e_n \notin U_1 \cup \ldots \cup U_n$. Thus, there must be a finite subcover. ///

The argument for the previous claim can be improved, to show

[6.3] Claim: In complete metric spaces, sequentially compact sets are compact.

[6.4] Remark: Again,

Proof: Let $\{U_\alpha : \alpha \in A\}$ be an open cover of a subset E of a complete metric space X, admitting no finite subcover. Using an equivalent of the Axiom of Choice, we can arrange to have a *minimal* subcover, that is, so that no U_β can be removed an still cover E. We do this at the end of the argument.

Granting this, without loss of generality the open cover is *minimal*, and not finite. Using the minimality (and again using the Axiom of Choice), for each index $\beta \in A$, let x_β be a point in E that is *not* in $\bigcup_{\alpha \neq \beta} U_\alpha$. Since the cover is minimal, these x_β's must be *distinct*. Since the cover is not finite, there are infinitely-many (distinct) x_β's. Since the are distinct, any countable subset of $\{x_\beta : \beta \in A\}$ gives a sequence $y_1, y_2, \ldots$ of distinct points. By sequential compactness, this sequence has at least one accumulation point $y_\infty \in E$.

Let U_{α_o} be an open in the cover containing y_∞. Since $\lim_n y_n = y_\infty$, there is n_o such that for all $n \geq n_o$ we have $y_n \in U_{\alpha_o}$. All those y_n's are among the x_β's, but the only x_β in U_{α_o} is x_{α_o}. That is, there cannot be infinitely-many distinct x_β's in U_{n_o}. Thus, assuming that a minimal cover is infinite leads to a contradiction.

To obtain a minimal subcover from a given cover $\{U_\alpha : \alpha \in A\}$, *well-order* the index set A. We choose a minimal subcover by transfinite induction, as follows. The idea is to ask, in the order chosen for A, cumulatively, whether or not U_α can be removed from the current subcover while still having a cover of the given set. That is, we inductively define a subset B of the index set A by transfinite induction: initially, $B = A$. At the α^{th} stage, remove α from B if U_α is unnecessary for maintaining the cover property. That is, remove α if

$$E \;\subset\; \bigcup_{\beta<\alpha,\ \beta\in B} U_\beta \;\cup\; \bigcup_{\beta>\alpha} U_\beta$$

otherwise keep α in B. By transfinite induction, B is an index set for a subcover of $\{U_\alpha : \alpha \in A\}$, and that subcover is *minimal* in the sense that no open can be removed without the result failing to be a cover.
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7. *Total-boundedness criterion for compact closure*

In general metric spaces, closed and bounded sets need not be compact (nor sequentially compact). More is required, as follows.

A set E in a metric space is *totally bounded* if, given $\varepsilon > 0$, there are finitely-many open balls of radius ε covering E. The property of *total boundedness* in a metric space is generally stronger than mere *boundedness.* It is immediate that any subset of a totally bounded set is totally bounded.

[7.1] Theorem: A set E in a metric space X has compact closure *if and only if* it is totally bounded.

[7.2] Remark: Sometimes a set with compact closure is said to be *pre-compact*.

Proof: Certainly if a set has compact closure then it admits a finite covering by open balls of arbitrarily small (positive) radius, by the compactness.

On the other hand, suppose that a set E is totally bounded in a complete metric space X. To show that E has compact closure it suffices to show *sequential* compactness, namely, that any sequence $\{x_i\}$ in E has a convergent subsequence.

We choose such a subsequence as follows. Cover E by finitely-many open balls of radius 1, invoking the total boundedness. In at least one of these balls there are infinitely-many elements from the sequence. Pick such a ball B_1, and let i_1 be the smallest index so that x_{i_1} lies in this ball.

The set $E \cap B_1$ is still totally bounded (and contains infinitely-many elements from the sequence). Cover it by finitely-many open balls of radius $1/2$, and choose a ball B_2 with infinitely-many elements of the sequence lying in $E \cap B_1 \cap B_2$. Choose the index i_2 to be the smallest one so that both $i_2 > i_1$ and so that x_{i_2} lies inside $E \cap B_1 \cap B_2$.

Proceeding inductively, suppose that indices $i_1 < \ldots < i_n$ have been chosen, and balls B_i of radius $1/i$, so that

$$x_i \in E \cap B_1 \cap B_2 \cap \ldots \cap B_i$$

Then cover $E \cap B_1 \cap \ldots \cap B_n$ by finitely-many balls of radius $1/(n+1)$ and choose one, call it B_{n+1}, containing infinitely-many elements of the sequence. Let i_{n+1} be the first index so that $i_{n+1} > i_n$ and so that

$$x_{n+1} \; \in \; E \cap B_1 \cap \ldots \cap B_{n+1}$$

Then for $m < n$ we have $d(x_{i_m}, x_{i_n}) \; \leq \; \frac{1}{m}$ so this subsequence is Cauchy. ///

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8. *Baire's theorem*

This standard result is both indispensable and mysterious.

A set E in a topological space X is *nowhere dense* if its closure $\bar{E}$ contains no non-empty open set. A *countable union* of nowhere dense sets is said to be *of first category*, while every other subset (if any) is *of second category*. The idea (not at all clear from this traditional terminology) is that first category sets are *small*, while second category sets are *large*. In this terminology, the theorem's assertion is equivalent to the assertion that (non-empty) *complete metric* spaces and *locally compact Hausdorff* spaces are *of second category*.

A G_δ set is a countable intersection of open sets. Concommitantly, an F_σ set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, *a countable intersection of dense G_δ's is still a dense G_δ.*

[8.1] Theorem: *(Baire)* Let X be either a complete metric space or a locally compact Hausdorff topological space. The intersection of a *countable* collection $U_1, U_2, \ldots$ of *dense open subsets* U_i of X is still *dense* in X.

Proof: Let B_o be a non-empty open set in X, and show that $\bigcap_i U_i$ meets B_o. Suppose that we have inductively chosen an open ball B_{n-1}. By the denseness of U_n, there is an open ball B_n whose closure $\overline{B_n}$ satisfies

$$\overline{B_n} \subset B_{n-1} \cap U_n$$

Further, for complete metric spaces, take B_n to have radius less than $1/n$ (or any other sequence of reals going to 0), and in the locally compact Hausdorff case take B_n to have compact closure.

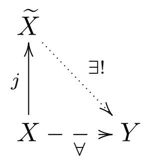
Let

$$K \;=\; \bigcap_{n\ge 1} \overline{B_n} \;\subset\; B_o \cap \bigcap_{n\ge 1} U_n$$

For complete metric spaces, the centers of the nested balls B_n form a Cauchy sequence (since they are nested and the radii go to 0). By completeness, this Cauchy sequence *converges*, and the limit point lies inside each *closure* $\overline{B_n}$, so lies in the intersection. In particular, K is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so K is non-empty, and B_o necessarily meets the intersection of the U_n. ///

9. *Appendix: mapping-property characterization of completion*

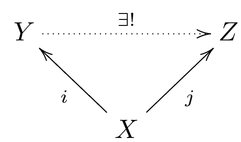
Our *intention* is that, when a metric space X is not complete, there should be a *complete* metric space $\widetilde{X}$ and an *isometry* (distance-preserving) $j : X \to \widetilde{X}$, such that every isometry $f : X \to Y$ to *complete* metric space Y *factors through* j uniquely. That is, there are *commutative diagrams*[7] of continuous maps



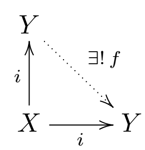
(for every isometry $X \to Y$)

Without describing any *constructions* of completions, we can prove some things about the behavior of *any possible* completion. In particular, we prove that any two completions are *naturally isometrically isomorphic* to each other. *Thus, the outcome will be independent of construction.*

[9.1] Claim: *(Uniqueness)* Let $i : X \to Y$ and $j : X \to Z$ be two completions of a metric space X. Then there is a unique isometric homeomorphism $h : Y \to Z$ such that $j = h \circ i$. That is, we have a commutative diagram

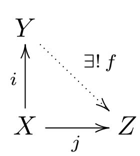


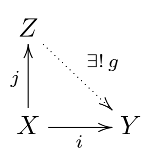
Proof: First, take $Y = Z$ and $f : X \to Y$ to be the inclusion i, in the characterization of $i : X \to Y$. The characterization of $i : X \to Y$ shows that there is *unique* isometry $f : Y \to Y$ fitting into a commutative diagram



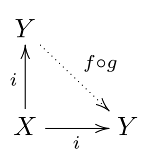
Since the *identity* map $Y \to Y$ certainly fits into this diagram, the *only* map f fitting into the diagram is the identity on Y.

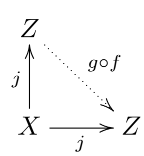
Next, applying the characterizations of both $i : X \to Y$ and $j : X \to Z$, we have unique $f : Y \to Z$ and $g : Z \to Y$ fitting into





Then $f \circ g : Y \to Y$ and $g \circ f : Z \to Z$ fit into





[7] A diagram of maps is *commutative* when the composite map from one object to another within the diagram does not depend on the route taken within the diagram.

By the first observation, this means that $f \circ g$ is the identity on Y, and $g \circ f$ is the identity on Z, so f and g are mutual inverses, and Y and Z are *homeomorphic*. ///

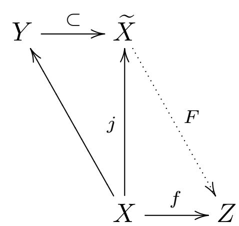
[9.2] Remark: A virtue of the characterization of completion is that it does not refer to the *internals* of any completion.

Next, we see that the mapping-property characterization of a completion does not introduce more points than absolutely necessary:

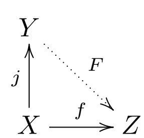
[9.3] Claim: Every point in a completion $\widetilde{X}$ of X is the limit of a Cauchy sequence in X. That is, X is *dense* in $\widetilde{X}$.

Proof: Write $d(,)$ for both the metric on X and its extension to $\widetilde{X}$. Let $Y \subset \widetilde{X}$ be the collection of limits of Cauchy sequences of points in X. We claim that Y itself is *complete*. Indeed, given a Cauchy sequence $\{y_i\}$ in Y with limit $z \in \widetilde{X}$, let $x_i \in X$ such that $d(x_i, y_i) < 2^{-i}$. It will suffice to show that $\{x_i\}$ is Cauchy with limit z. Indeed, given $\varepsilon > 0$, take N large enough so that $d(y_i, z) < \varepsilon/2$ for all $i \geq N$, and increase N if necessary so that $2^{-i} < \varepsilon/2$. Then, by the triangle inequality, $d(x_i, z) < \varepsilon$ for all $i \geq N$. Thus, Y is complete.

By the defining property of $\widetilde{X}$, every isometry $f : X \to Z$ to complete Z has a unique extension to an isometry $F : \widetilde{X} \to Z$ fitting into



Since Y is already complete and $j(X) \subset Y$, the restriction of F to Y gives a diagram



That is, Y fits the characterization of a completion of X. By uniqueness, $Y \subset \widetilde{X}$ is a homeomorphism, so $Y = \widetilde{X}$. ///