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# 02. *Basic measure theory*

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## 1. *Borel-measurable functions and pointwise limits*

Pointwise limits of continuous functions on $\mathbb{R}$ or on intervals $[a, b]$ need not be continuous. We want a class of functions closed under taking pointwise limits of sequences. The following is the simplest form of a general discussion.

The collection of *Borel subsets* of $\mathbb{R}$ is the smallest collection of subsets of $\mathbb{R}$ closed under taking *countable unions*, under *countable intersections*, under *complements*, and containing all open and closed subsets of $\mathbb{R}$. This is also called the Borel $\sigma$-*algebra* in $\mathbb{R}$. To be sure that this description makes sense, we prove:

**[1.1] Claim:** Intersections of $\sigma$-algebras of subsets of $\mathbb{R}$ are $\sigma$-algebras. Thus, the *smallest* $\sigma$-algebra containing a given set of sets is the intersection of all $\sigma$-algebras containing it.

*Proof:* Let $S$ be a set of subsets of a set $X$, and $\{A_i : i \in I\}$ a collection of $\sigma$-algebras containing $S$. Let $A$ be the intersection $\bigcap_i A_i$. Given a countable collection $E_1, E_2, \ldots$ of sets in $A$, for every $i \in I$ the set $E_j$ are in $A_i$, so their intersection and union are in $A_i$. Since this holds for every $i \in I$, that intersection and union are in $A$. The argument for complements is even simpler. ///

There is traditional terminology for certain simple types of Borel sets. For example a *countable intersection of open sets* is a $G_\delta$ set, while a *countable union of closed sets* is an $F_\sigma$. The notation can be iterated: a $G_{\delta\sigma}$ is a countable union of countable intersections of opens, and so on. We will not need this.

A simple useful choice of larger class of functions than continuous is: a real-valued or complex-valued function $f$ on $\mathbb{R}$ is *Borel-measurable* when the inverse image $f^{-1}(U)$ is a Borel set for every open set $U$ in the target space.

First, we verify some immediate desirable properties:

**[1.2] Claim:** The sum and product of two Borel-measurable functions are Borel-measurable. For non-vanishing Borel-measurable $f$, $1/f$ is Borel-measurable.

*Proof:* As a warm-up to this argument, it is useful to rewrite the $\varepsilon - \delta$ proof, that the sum of two continuous functions is continuous, in terms of the condition that inverse images of opens are open.

For Borel-measurable $f, g$ on $\mathbb{R}$, let $f \oplus g$ be the $\mathbb{R} \times \mathbb{R}$-valued function on $\mathbb{R} \times \mathbb{R}$ defined by $(f \oplus g)(x, y) = (f(x), g(y))$. Let $s : \mathbb{R} \times \mathbb{R}$ be the sum map, $s(x, y) = x + y$. Let $\Delta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the diagonal map

$\Delta(x) = (x, x)$. Both $s$ and $\Delta$ are continuous, and

$$(f+g)^{-1} \;=\; \Delta^{-1} \circ (f \oplus g)^{-1} \circ s^{-1}$$

Since $s$ is continuous, for open $U \subset \mathbb{R}$, $s^{-1}(U)$ is open in $\mathbb{R} \times \mathbb{R}$, and is a countable union of open rectangles $(a_i, b_i) \times (c_i, d_i)$. Then

$$(f \oplus g)^{-1}(s^{-1}(U)) \;=\; \bigcup_i (f \oplus g)^{-1}((a_i, b_i) \times (c_i, d_i)) \;=\; \bigcup_i f^{-1}(a_i, b_i) \times g^{-1}(c_i, d_i)$$

and every inverse image $f^{-1}(a_i, b_i)$ and $g^{-1}(c_i, d_i)$ is *Borel measurable.* Then

$$\Delta^{-1}\Big(f^{-1}(a_i, b_i) \times g^{-1}(c_i, d_i)\Big) \;=\; f^{-1}(a_i, b_i) \cap g^{-1}(c_i, d_i)\Big) \;=\; (\text{Borel measurable})$$

The countable union indexed by $i$ is still Borel-measurable, so $(f+g)^{-1}(U)$ is measurable. The arguments for product and inverse are nearly identical, since product and inverse (away from 0) are continuous. ///

It is sometimes useful to allow the target space for functions to be the *two-point compactification* $Y = \{-\infty\} \cup \mathbb{R} \cup +\infty$ of the real line, with neighborhood basis $-\infty \cup (-\infty, a)$ at $-\infty$ and $(a, +\infty) \cup \{+\infty\}$ at $+\infty$ when we need to allow functions to blow up in some fashion. But $\pm\infty$ are not numbers, and do not admit consistent manipulation as though they were.

A more serious positive indicator of the reasonable-ness of Borel-measurable functions as a larger class containing continuous functions:

**[1.3] Theorem:** Every pointwise limit of Borel-measurable functions is Borel-measurable. More generally, every countable *inf* and countable *sup* of Borel-measurable functions is Borel-measurable, as is every countable *liminf* and *limsup.*

*Proof:* We prove that a countable $f(x) = \inf_n f_n(x)$ is measurable. Observe that $f(x) < b$ if and only if there is some $n$ such that $f_n(x) < b$. Thus,

$$f^{-1}(-\infty, b) \;=\; \bigcup_n f_n^{-1}(-\infty, b) \;=\; (\text{countable union of measurables}) \;=\; (\text{measurable})$$

Further,

$$f^{-1}(-\infty, a] \;=\; \bigcap_n f^{-1}(-\infty, a + \frac{1}{n}) \;=\; (\text{countable intersection of measurables}) \;=\; (\text{measurable})$$

and then

$$f^{-1}(a, b) \;=\; f^{-1}(-\infty, b) - f^{-1}(-\infty, a] \;=\; f^{-1}(-\infty, b) \cap (\mathbb{R} - f^{-1}(-\infty, a])$$

$$= (\text{intersection of measurable with complement of measurable}) \;=\; (\text{measurable})$$

A nearly identical argument proves measurability of countable *sup*s of measurable functions.

A slight enhancement of this argument treats *liminfs* and *limsups*: $\limsup_n f_n(x) < b$ if and only if, for all $n_o$, there is $n \geq n_o$ such that $f_n(x) < b$:

$$\{x : \liminf_n f_n(x) < b\} \;=\; \bigcap_{n \geq 1} \Big( \bigcup_{n \geq n_o} f_n^{-1}(-\infty, b)\Big)$$

$$= (\text{countable intersection of countable unions of measurables}) \;=\; (\text{measurable})$$

The rest of the argument for measurability of pointwise *liminfs* is identical to that for *infs*, and also for *limsups*. When pointwise $\lim_n f_n(x)$ exists, it is $\liminf_n f_n(x)$, showing that countable limits of measurable are measurable. ///

## 2. *Lebesgue-measurable functions and almost-everywhere pointwise limits*

A sequence $\{f_n\}$ of Borel-measurable functions on $\mathbb{R}$ converges (pointwise) *almost everywhere* when there is a Borel set $N \subset \mathbb{R}$ of measure 0 such that $\{f_n\}$ converges pointwise on $\mathbb{R} - N$. One of Lebesgue's discoveries was that ignoring what may happen on sets of measure zero was an essential simplifying point in many situations.

However, there are sets of Lebesgue measure 0 that are not Borel sets. Thus, *almost-everywhere* pointwise limits of Borel-measurable functions may fall into a larger class. That is, there is a larger $\sigma$-algebra than that of Borel sets. Indeed, the description of the Lebesgue (outer) measure suggests that *any subset $F$ of a Borel set $E$ of measure zero should itself be measurable, with measure zero.*

The smallest $\sigma$-algebra containing all Borel sets in $\mathbb{R}$ and containing all subsets of Lebesgue-measure-zero Borel sets is the $\sigma$-algebra of *Lebesgue-measurable* sets in $\mathbb{R}$.

**[2.1] Claim:** Finite sums, finite products, and inverses (of non-zero) Lebesgue-measurable functions are Lebesgue-measurable.

*Proof:* The proofs in the previous section did not use any specifics of the $\sigma$-algebra of Borel-measurable functions, so the same proofs succeed. ///

**[2.2] Theorem:** Every pointwise-almost-everywhere limit of Lebesgue-measurable functions $f_n$ is Lebesgue-measurable.

*Proof:* Again, the proofs in the previous section did not use any specifics of the $\sigma$-algebra of Borel-measurable functions. ///

## 3. *Borel measures*

A *Borel measure* $\mu$ is an assignment of (often *non-negative*) real numbers $\mu(E)$ (measures) to Borel sets $E$, in a fashion that is *countably additive* for disjoint unions:

$$\mu(E_1 \cup E_2 \cup E_3 \cup \ldots) \;=\; \mu(E_1) + \mu(E_2) + \mu(E_3) + \ldots \qquad \text{(for } \textit{disjoint} \text{ Borel sets } E_1, E_2, E_3, \ldots)$$

The most important prototype of a Borel measure is *Lebesgue (outer) measure* of a Borel set $E \subset \mathbb{R}$, described by

$$\mu(E) \;=\; \inf\{\sum_{n=1}^{\infty} |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

That is, it is the *inf* of the sums of lengths of the intervals in a countable cover of $E$ by open intervals. For example, any countable set has (Lebesgue) measure 0.

That is, there is a $\sigma$-algebra $A$ including Borel sets (equivalently, including open sets), and $\mu$ is a (often non-negative real-valued) function on $A$ with the countable additivity above.

*[... iou ...]*

**[3.1] Remark:** Assuming the Axiom of Choice, one can prove that there is no Borel measure $\mu$ with $\sigma$-algebra containing *all* subsets of $\mathbb{R}$. So our ambitions for assigning measures should be more modest.

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# 4. *Lebesgue integrals*

With such notion of *measure*, there is a corresponding *integrability* and *integral*, due to Lebesgue. It amounts to replacing the literal rectangles used in Riemann integration by more general rectangles, with bases not just intervals, but measurable sets, as follows.

The *characteristic function* or *indicator function* $\mathrm{ch}_E$ or $\chi_E$ of a measurable subset $E \subset \mathbb{R}$ is 1 on $E$ and 0 off. A *simple function* is a finite, positive-coefficiented, linear combination of characteristic functions of bounded measurable sets, that is, is of the form

$$\text{(simple function) } s \;=\; \sum_{i=1}^{n} c_i \cdot \mathrm{ch}_{E_i} \qquad\qquad \text{(with } c_i \geq 0\text{)}$$

The *integral* of $s$ is what one would expect:

$$\int s\, d\mu \;=\; \int \Big(\sum_{i=1}^{n} c_i \cdot \mathrm{ch}_{E_i}\Big)\, d\mu \;=\; \sum_i c_i \cdot \mu(E_i)$$

Next, the measure of a *non-negative* function $f$ is the *sup* of the integrals of all simple functions between $f$ and 0:

$$\int f\, d\mu \;=\; \sup_{0 \leq s \leq f} \int s\, d\mu \qquad\qquad \text{(sup over simple } s \text{ with } 0 \leq s(x) \leq f(x) \text{ for all } x\text{)}$$

After proving that the positive and negative parts $f_+$ and $f_-$ of Borel measurable real-valued $f$ are again Borel measurable,

$$\int f\, d\mu \;=\; \int f_+\, d\mu - \int (-f_-)\, d\mu$$

Similarly, for complex-valued $f$, break $f$ into real and imaginary parts.

There are details to be checked:

**[4.1] Theorem:** Borel-measurable functions $f, g$ taking values in $[0, +\infty]$ are *integrable*, in the sense that the previous prescription yields an assignment $f \to \int_{\mathbb{R}} f \in [0, +\infty]$ such that for positive constants $a, b$

$$\int_{\mathbb{R}} (af + bg) \;=\; a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g \qquad\qquad \text{(for all } a, b \geq 0\text{)}$$

For complex-valued Borel-measurable $f, g$, the absolute values $|f|$ and $|g|$ are Borel-measurable. Assuming $\int_{\mathbb{R}} |f| < \infty$ and $\int_{\mathbb{R}} |g| < \infty$, for any complex $a, b$

$$\int_{\mathbb{R}} (af + bg) \;=\; a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$$

*Proof:* [... iou ...] ///

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# 5. *Convergence theorems: monotone, dominated*

Easy, natural examples show that *pointwise* limits $f = \lim_n f_n$ of measurable functions $f_n$, while still measurable, need *not* satisfy $\int f = \lim \int f_n$. That is, this failure is not a pathology, but, rather, is completely reasonable. Hence additional conditions are essential to know that the integral of a pointwise limit is the limit of the integrals.

First, a relatively simple initial step:

**[5.1] Theorem:** *(Fatou's lemma)* For Borel-measurable $f_n$ with values in $[0, +\infty]$, the pointwise $f(x) = \liminf_n f_n(x)$ is Borel-measurable, and

$$\int \liminf_n f_n(x)\, dx \;\le\; \liminf_n \int f_n$$

*Proof:* [... iou ...] ///

**[5.2] Theorem:** *(Lebesgue: monotone convergence)* Let $f_1, f_2, \ldots$ be a sequence of non-negative real-valued Lebesgue-measurable functions on $[a, b]$, with $f_1(x) \le f_2(x) \le \ldots$ for all $x$. Then $\int_a^b \lim_n f_n(x)\, dx = \lim_n \int_a^b f_n(x)\, dx$. This includes the possibility that some of the limits of the pointwise values are $+\infty$, and that the integral of the limit is $+\infty$.

*Proof:* [... iou ...] ///

**[5.3] Theorem:** *(Lebesgue: dominated convergence)* Let $f_1, f_2, \ldots$ be a sequence of complex-valued Lebesgue-measurable functions on $[a, b]$, with $|f_n(x)| \le g(x)$ for all $x$, for some measurable $g$ with $\int_a^b g(x)\, dx < +\infty$. Then $\int_a^b \lim_n f_n(x)\, dx = \lim_n \int_a^b f_n(x)\, dx$.

*Proof:* [... iou ...] ///

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# 6. *Urysohn's lemma*

Urysohn's lemma proves existence of sufficiently many functions on reasonable topological spaces.

**[6.1] Theorem:** *(Urysohn)* In a locally compact Hausdorff topological space $X$, given a compact subset $K$ contained in an open set $U$, there is a continuous function $0 \le f \le 1$ which is 1 on $K$ and 0 off $U$.

*Proof:* First, we prove that there is an open set $V$ such that

$$K \;\subset\; V \;\subset\; \overline{V} \;\subset\; U$$

For each $x \in K$ let $V_x$ be an open neighborhood of $x$ with compact closure. By compactness of $K$, some finite subcollection $V_{x_1}, \ldots, V_{x_n}$ of these $V_x$ cover $K$, so $K$ is contained in the open set $W = \bigcup_i V_{x_i}$ which has compact closure $\bigcup_i \overline{V}_{x_i}$ since the union is *finite*.

Using the compactness again in a similar fashion, for each $x$ in the closed set $X - U$ there is an open $W_x$ containing $K$ and a neighborhood $U_x$ of $x$ such that $W_x \cap U_x = \phi$.

Then

$$\bigcap_{x \in X-U} (X - U) \cap \overline{W} \cap \overline{W}_x \;=\; \phi$$

These are compact subsets in a Hausdorff space, so (again from compactness) some *finite* subcollection has empty intersection, say

$$(X - U) \cap \big(\overline{W} \cap \overline{W}_{x_1} \cap \ldots \cap \overline{W}_{x_n}\big) \;=\; \phi$$

That is,

$$\overline{W} \cap \overline{W}_{x_1} \cap \ldots \cap \overline{W}_{x_n} \;\subset\; U$$

Thus, the open set

$$V \;=\; W \cap W_{x_1} \cap \ldots \cap W_{x_n}$$

meets the requirements.

Using the possibility of inserting an open subset and its closure between any $K \subset U$ with $K$ compact and $U$ open, we inductively create opens $V_r$ (with compact closures) indexed by rational numbers $r$ in the interval $0 \le r \le 1$ such that, for $r > s$,

$$K \subset V_r \subset \overline{V}_r \subset V_s \subset \overline{V}_s \subset U$$

From any such configuration of opens we construct the desired continuous function $f$ by

$$f(x) = \sup\{r \text{ rational in } [0,1] : x \in V_r, \} = \inf\{r \text{ rational in } [0,1] : x \in \overline{V}_r, \}$$

It is not immediate that this sup and inf are the same, but if we *grant* their equality then we can prove the *continuity* of this function $f(x)$. Indeed, the sup description expresses $f$ as the supremum of characteristic functions of open sets, so $f$ is at least *lower semi-continuous.* [1] The inf description expresses $f$ as an infimum of characteristic functions of closed sets so is *upper* semi-continuous. Thus, $f$ would be continuous.

To finish the argument, we must construct the sets $V_r$ and prove equality of the inf and sup descriptions of the function $f$.

To construct the sets $V_i$, start by finding $V_0$ and $V_1$ such that

$$K \subset V_1 \subset \overline{V}_1 \subset V_0 \subset \overline{V}_0 \subset U$$

Fix a well-ordering $r_1, r_2, \ldots$ of the rationals in the open interval $(0,1)$. Supposing that $V_{r_1}, \ldots, v_{r_n}$ have been chosen. let $i, j$ be indices in the range $1, \ldots, n$ such that

$$r_j > r_{n+1} > r_i$$

and $r_j$ is the *smallest* among $r_1, \ldots, r_n$ *above* $r_{n+1}$, while $r_i$ is the *largest* among $r_1, \ldots, r_n$ *below* $r_{n+1}$. Using the first observation of this argument, find $V_{r_{n+1}}$ such that

$$V_{r_j} \subset \overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_i} \subset \overline{V}_{r_i}$$

This constructs the nested family of opens.

Let $f(x)$ be the sup and $g(x)$ the inf of the characteristic functions above. If $f(x) > g(x)$ then there are $r > s$ such that $x \in V_r$ and $x \notin \overline{V}_s$. But $r > s$ implies that $V_r \subset \overline{V}_s$, so this cannot happen. If $g(x) > f(x)$, then there are rationals $r > s$ such that

$$g(x) > r > s > f(x)$$

Then $s > f(x)$ implies that $x \notin V_s$, and $r < g(x)$ implies $x \in \overline{V}_r$. But $V_r \subset \overline{V}_s$, contradiction. Thus, $f(x) = g(x)$. ///

# 7. *Comparison to continuous functions: Lusin's theorem*

One aspect of the following theorem is that we have not inadvertently needlessly included functions wildly unrelated to continuous functions:

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[1] A (real-valued) function $f$ is *lower* semi-continuous when for all bounds $B$ the set $\{x : f(x) > B\}$ is open. The function $f$ is *upper* semi-continuous when for all bounds $B$ the set $\{x : f(x) < B\}$ is open. It is easy to show that a sup of lower semi-continuous functions is lower semi-continuous, and an inf of upper semi-continuous functions is upper semi-continuous. As expected, a function both upper and lower semi-continuous is continuous.

**[7.1] Theorem:** *(Lusin)* Continuous functions approximate Borel-measurable functions well: given Borel-measurable real-valued or complex-valued $f$ on $\mathbb{R}$, for every $\varepsilon > 0$ and for every Borel subset $\Omega \subset \mathbb{R}$ of finite Lebesgue measure, there is a relative closed $E \subset \Omega$ such that $\mu(\Omega - E) < \varepsilon$, and $f|_E$ is *continuous*.

*Proof:* [... iou ...] ///

Not much better can be done than Lusin's theorem says: for example, continuous approximations to the Heaviside step function

$$H(x) \;=\; \begin{cases} 0 & \text{for } x < 0 \\ \\ 1 & \text{for } x \geq 0 \end{cases}$$

have to go from 0 to 1 *somewhere*, by the Intermediate Value Theorem, so will be in $(\frac{1}{4}, \frac{3}{4})$ on an open set of strictly positive measure.

**[7.2] Remark:** It turns out that the everyday use of measure theory, measurable functions, and so on, does *not* proceed by way of Lusin's theorem or similar direct connections with continuous functions, but, rather, by direct interaction with the more general ideas.

# 8. *Comparison to uniform pointwise convergence: Severini-Egoroff*

**[8.1] Theorem:** *(Severini, Egoroff)* Pointwise convergence of sequences of Borel-measurable functions is approximately *uniform* convergence: given a almost-everywhere pointwise-convergent sequence $\{f_n\}$ of Borel-measurable functions on $\mathbb{R}$, for every $\varepsilon > 0$ and for every Borel subset $\Omega \subset \mathbb{R}$ of finite Lebesgue measure, there is a Borel subset $E \subset \Omega$ such that $\{f_n\}$ converges *uniformly* pointwise on $E$.

*Proof:* [... iou ...] ///

**[8.2] Remark:** Despite the connection that the Severini-Egoroff theorem makes between pointwise and *uniform* pointwise convergence, this idea turns out *not* to be the way to understand convergence of measurable functions. Instead, the game becomes ascertaining additional conditions that guarantee convergence of integrals, as earlier.

# 9. *Integration on products: Fubini-Tonelli*

*[... iou ...]*

More interesting, and more useful: after figuring out how to characterize measure on product spaces,

**[9.1] Theorem:** *(Fubini-Tonelli)* For complex-valued measurable $f, g$, if any one of $\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x,y)|\; dx\, dy$, $\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x,y)|\; dy\, dx$, or $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\; d\mathrm{vol}$ is finite, then the all are finite, and are equal. For $[0, +\infty]$-valued functions $f$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\; dx\, dy \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\; dy\, dx \;=\; \int_{\mathbb{R}\times\mathbb{R}} f(x,y)\; d\mathrm{vol}$$

although the values may be $+\infty$.

*Proof:* [... iou ...] ///