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06. Product measures and Fubini-Tonelli theorem

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1. *Product measures, completions of measures*

Let X, μ and Y, ν be measure spaces with corresponding σ-algebras A, B. Assume X and Y are σ-*finite*, in the sense that they are countable unions of finite-measure sets.

First, the *product* σ-algebra is the σ-algebra in $X \times Y$ generated by all products $E \times F$ with $E \in A$ and $F \in B$.

For iterated integrals to make sense, we need to check a few things. For $E \in A \times B$, for $x \in X$ and $y \in Y$, let

$$E_x \;=\; \{y \in Y : (x, y) \in E\} \qquad \text{and} \qquad E^y \;=\; \{x \in X : (x, y) \in E\}$$

As a consistency check, we have

[1.1] Theorem: For $E \in A \times B$, for $x \in X$ and $y \in Y$, $E_x \in A$ and $E^y \in B$. The function $x \to \nu(E_x)$ is μ-measurable, $y \to \mu(E^y)$ is ν-measurable, and

$$\int_X \nu(E_x)\, d\mu(x) \;=\; \int_Y \mu(E^y)\, d\nu(y)$$

Proof: [... iou ...] ///

Then the *product measure* $\mu \times \nu$ can be defined in the expected fashion: for $E \in A \times B$,

$$(\mu \times \nu)(E) \;=\; \int_X \nu(E_x)\, d\mu(x) \;=\; \int_Y \mu(E^y)\, d\nu(y)$$

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2. *Fubini-Tonelli theorem(s)*

Let X, μ and Y, ν be measure spaces with corresponding σ-algebras A, B. Assume X and Y are σ-*finite*.

[2.1] Theorem: *(Fubini-Tonelli)* For complex-valued measurable f, g, if any one of

$$\int_X \int_Y |f(x,y)|\, d\mu(x)\, d\nu(y) \qquad \int_Y \int_X |f(x,y)|\, d\nu(y)\, d\mu(x) \qquad \int_{X\times Y} |f(x,y)|\, d\mu{\times}\nu$$

is finite, then they *all* are finite, and are equal. For $[0, +\infty]$-valued functions f,

$$\int_X \int_Y f(x,y)\, d\mu(x)\, d\nu(y) \;=\; \int_Y \int_X f(x,y)\, d\nu(y)\, d\mu(x) \;=\; \int_{X\times Y} f(x,y)\, d\mu{\times}\nu$$

although the values may be $+\infty$.

Proof: [... iou ...] ///

sectionCompletions of measures

For example, a reasonable measure on $\mathbb{R}^m \times \mathbb{R}^n$ should include many sets not expressible as countable unions of products $E \times F$ where $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^n$. For example, diagonal subsets of the form $D = \{(x,x) : 0 \le x \le 1\} \subset \mathbb{R}^2$ are not countable unions of products, but should surely be measurable.

One way to accomplish this is by *completion* of the product measure.

Then the *completion* of $\mu \times \nu$ further assigns measure 0 to *any* subset S of $T \in A \times B$ with $(\mu \times \nu)(T) = 0$, and adjoins all such sets to the σ-algebra $A \times B$.

[2.2] Claim: Lebesgue measure on $\mathbb{R}^m \times \mathbb{R}^n$ is the completion of the product of Lebesgue measures on $\mathbb{R}^m$ and $\mathbb{R}^n$.

Proof: [... iou ...] ///

Completing a product measure is usually what we want, but it slightly complicates the statement of the corresponding Fubini-Tonelli theorem:

[2.3] Theorem: Let X, A, μ and Y, B, ν be *complete* measure spaces, with X, Y σ-finite. Let f be a function on $X \times Y$ measurable with respect to the *completion* of the product measure. Then $x \to f(x,y)$ and $y \to f(x,y)$ are μ-measurable and ν-measurable (only) *almost everywhere*.

Proof: [... iou ...] ///

[2.4] Remark: To be precise, *completeness* is a property of σ-algebras, not of measures.
