

(October 3, 2017)

02. Measure and integral

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2017-18/02_measure_and_integral.pdf]

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1. Borel-measurable functions and pointwise limits

Pointwise limits of continuous functions on \mathbb{R} or on intervals $[a, b]$ need not be continuous. We want a class of functions closed under taking pointwise limits of sequences. The following is the simplest form of a general discussion.

The collection of *Borel subsets* of \mathbb{R} is the smallest collection of subsets of \mathbb{R} closed under taking *countable unions*, under *countable intersections*, under *complements*, and containing all open and closed subsets of \mathbb{R} . This is also called the Borel σ -algebra in \mathbb{R} . To be sure that this description makes sense, we prove:

[1.1] **Claim:** Intersections of σ -algebras of subsets of \mathbb{R} are σ -algebras. Thus, the *smallest* σ -algebra containing a given set of sets is the intersection of all σ -algebras containing it.

Proof: Let S be a set of subsets of a set X , and $\{A_i : i \in I\}$ a collection of σ -algebras containing S . Let A be the intersection $\bigcap_i A_i$. Given a countable collection E_1, E_2, \dots of sets in A , for every $i \in I$ the set E_j are in A_i , so their intersection and union are in A_i . Since this holds for every $i \in I$, that intersection and union are in A . The argument for complements is even simpler. ///

There is traditional terminology for certain simple types of Borel sets. For example a *countable intersection of open sets* is a G_δ set, while a *countable union of closed sets* is an F_σ . The notation can be iterated: a $G_{\delta\sigma}$ is a countable union of countable intersections of opens, and so on. We will not need this.

A simple useful choice of larger class of functions than continuous is: a real-valued or complex-valued function f on \mathbb{R} is *Borel-measurable* when the inverse image $f^{-1}(U)$ is a Borel set for every open set U in the target space.

First, we verify some immediate desirable properties:

[1.2] **Claim:** The sum and product of two Borel-measurable functions are Borel-measurable. For non-vanishing Borel-measurable f , $1/f$ is Borel-measurable.

Proof: As a warm-up to this argument, it is useful to rewrite the $\varepsilon - \delta$ proof, that the sum of two continuous functions is continuous, in terms of the condition that inverse images of opens are open.

For Borel-measurable f, g on \mathbb{R} , let $f \oplus g$ be the $\mathbb{R} \times \mathbb{R}$ -valued function on $\mathbb{R} \times \mathbb{R}$ defined by $(f \oplus g)(x, y) = (f(x), g(y))$. Let $s : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the sum map, $s(x, y) = x + y$. Let $\Delta : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ be the diagonal map

$\Delta(x) = (x, x)$. Both s and Δ are continuous, and

$$(f + g)^{-1} = \Delta^{-1} \circ (f \oplus g)^{-1} \circ s^{-1}$$

Since s is continuous, for open $U \subset \mathbb{R}$, $s^{-1}(U)$ is open in $\mathbb{R} \times \mathbb{R}$, and is a countable union of open rectangles $(a_i, b_i) \times (c_i, d_i)$. Then

$$(f \oplus g)^{-1}(s^{-1}(U)) = \bigcup_i (f \oplus g)^{-1}((a_i, b_i) \times (c_i, d_i)) = \bigcup_i f^{-1}(a_i, b_i) \times g^{-1}(c_i, d_i)$$

and every inverse image $f^{-1}(a_i, b_i)$ and $g^{-1}(c_i, d_i)$ is *Borel measurable*. Then

$$\Delta^{-1}\left(f^{-1}(a_i, b_i) \times g^{-1}(c_i, d_i)\right) = f^{-1}(a_i, b_i) \cap g^{-1}(c_i, d_i) = (\text{Borel measurable})$$

The countable union indexed by i is still Borel-measurable, so $(f + g)^{-1}(U)$ is measurable. The arguments for product and inverse are nearly identical, since product and inverse (away from 0) are continuous.

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It is sometimes useful to allow the target space for functions to be the *two-point compactification* $Y = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ of the real line, with neighborhood basis $-\infty \cup (-\infty, a)$ at $-\infty$ and $(a, +\infty) \cup \{+\infty\}$ at $+\infty$ when we need to allow functions to blow up in some fashion. But $\pm\infty$ are not numbers, and do not admit consistent manipulation as though they were.

A more serious positive indicator of the reasonable-ness of Borel-measurable functions as a larger class containing continuous functions:

[1.3] Theorem: Every pointwise limit of Borel-measurable functions is Borel-measurable. More generally, every countable *inf* and countable *sup* of Borel-measurable functions is Borel-measurable, as is every countable *liminf* and *limsup*.

Proof: We prove that a countable $f(x) = \inf_n f_n(x)$ is measurable. Observe that $f(x) < b$ if and only if there is some n such that $f_n(x) < b$. Thus,

$$f^{-1}(-\infty, b) = \bigcup_n f_n^{-1}(-\infty, b) = (\text{countable union of measurables}) = (\text{measurable})$$

Further,

$$f^{-1}(-\infty, a] = \bigcap_n f^{-1}(-\infty, a + \frac{1}{n}) = (\text{countable intersection of measurables}) = (\text{measurable})$$

and then

$$\begin{aligned} f^{-1}(a, b) &= f^{-1}(-\infty, b) - f^{-1}(-\infty, a] = f^{-1}(-\infty, b) \cap (\mathbb{R} - f^{-1}(-\infty, a]) \\ &= (\text{intersection of measurable with complement of measurable}) = (\text{measurable}) \end{aligned}$$

A nearly identical argument proves measurability of countable *sup*s of measurable functions.

A slight enhancement of this argument treats *liminfs* and *limsups*: $\limsup_n f_n(x) < b$ if and only if, for all n_o , there is $n \geq n_o$ such that $f_n(x) < b$:

$$\begin{aligned} \{x : \liminf_n f_n(x) < b\} &= \bigcap_{n \geq 1} \left(\bigcup_{n \geq n_o} f_n^{-1}(-\infty, b) \right) \\ &= (\text{countable intersection of countable unions of measurables}) = (\text{measurable}) \end{aligned}$$

The rest of the argument for measurability of pointwise *liminfs* is identical to that for *infs*, and also for *limsup*s. When pointwise $\lim_n f_n(x)$ exists, it is $\liminf_n f_n(x)$, showing that countable limits of measurable are measurable. ///

2. Lebesgue-measurable functions and almost-everywhere pointwise limits

A sequence $\{f_n\}$ of Borel-measurable functions on \mathbb{R} converges (pointwise) *almost everywhere* when there is a Borel set $N \subset \mathbb{R}$ of measure 0 such that $\{f_n\}$ converges pointwise on $\mathbb{R} - N$. One of Lebesgue's discoveries was that ignoring what may happen on sets of measure zero was an essential simplifying point in many situations.

However, there are sets of Lebesgue measure 0 that are not Borel sets. Thus, *almost-everywhere* pointwise limits of Borel-measurable functions may fall into a larger class. That is, there is a larger σ -algebra than that of Borel sets. Indeed, the description of the Lebesgue (outer) measure suggests that *any subset F of a Borel set E of measure zero should itself be measurable, with measure zero*.

The smallest σ -algebra containing all Borel sets in \mathbb{R} and containing all subsets of Lebesgue-measure-zero Borel sets is the σ -algebra of *Lebesgue-measurable* sets in \mathbb{R} .

[2.1] **Claim:** Finite sums, finite products, and inverses (of non-zero) Lebesgue-measurable functions are Lebesgue-measurable.

Proof: The proofs in the previous section did not use any specifics of the σ -algebra of Borel-measurable functions, so the same proofs succeed. ///

[2.2] **Theorem:** Every pointwise-almost-everywhere limit of Lebesgue-measurable functions f_n is Lebesgue-measurable.

Proof: Again, the proofs in the previous section did not use any specifics of the σ -algebra of Borel-measurable functions. ///

3. Borel measures

A *Borel measure* μ is an assignment of (often *non-negative*) real numbers $\mu(E)$ (measures) to Borel sets E , in a fashion that is *countably additive* for disjoint unions:

$$\mu(E_1 \cup E_2 \cup E_3 \cup \dots) = \mu(E_1) + \mu(E_2) + \mu(E_3) + \dots \quad (\text{for disjoint Borel sets } E_1, E_2, E_3, \dots)$$

The most important prototype of a Borel measure is *Lebesgue (outer) measure* of a Borel set $E \subset \mathbb{R}$, described by

$$\mu(E) = \inf \left\{ \sum_{n=1}^{\infty} |b_n - a_n| : E \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}$$

That is, it is the *inf* of the sums of lengths of the intervals in a countable cover of E by open intervals. For example, any countable set has (Lebesgue) measure 0.

That is, there is a σ -algebra A including Borel sets (equivalently, including open sets), and μ is a (often non-negative real-valued) function on A with the countable additivity above.

[... iou ...]

[3.1] **Remark:** Assuming the Axiom of Choice, one can prove that there is no Borel measure μ with σ -algebra containing *all* subsets of \mathbb{R} . So our ambitions for assigning measures should be more modest.

4. Lebesgue integrals

With such notion of *measure*, there is a corresponding *integrability* and *integral*, due to Lebesgue. It amounts to replacing the literal rectangles used in Riemann integration by more general rectangles, with bases not just intervals, but measurable sets, as follows.

The *characteristic function* or *indicator function* ch_E or χ_E of a measurable subset $E \subset \mathbb{R}$ is 1 on E and 0 off. A *simple function* is a finite, positive-coefficiented, linear combination of characteristic functions of bounded measurable sets, that is, is of the form

$$\text{(simple function) } s = \sum_{i=1}^n c_i \cdot \text{ch}_{E_i} \quad (\text{with } c_i \geq 0)$$

The *integral* of s is what one would expect:

$$\int s \, d\mu = \int \left(\sum_{i=1}^n c_i \cdot \text{ch}_{E_i} \right) d\mu = \sum_i c_i \cdot \mu(E_i)$$

Next, the measure of a *non-negative* function f is the *sup* of the integrals of all simple functions between f and 0:

$$\int f \, d\mu = \sup_{0 \leq s \leq f} \int s \, d\mu \quad (\text{sup over simple } s \text{ with } 0 \leq s(x) \leq f(x) \text{ for all } x)$$

After proving that the positive and negative parts f_+ and f_- of Borel measurable real-valued f are again Borel measurable,

$$\int f \, d\mu = \int f_+ \, d\mu - \int (-f_-) \, d\mu$$

Similarly, for complex-valued f , break f into real and imaginary parts.

There are details to be checked:

[4.1] Theorem: Borel-measurable functions f, g taking values in $[0, +\infty]$ are *integrable*, in the sense that the previous prescription yields an assignment $f \rightarrow \int_{\mathbb{R}} f \in [0, +\infty]$ such that for positive constants a, b

$$\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g \quad (\text{for all } a, b \geq 0)$$

For complex-valued Borel-measurable f, g , the absolute values $|f|$ and $|g|$ are Borel-measurable. Assuming $\int_{\mathbb{R}} |f| < \infty$ and $\int_{\mathbb{R}} |g| < \infty$, for any complex a, b

$$\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$$

Proof: [... iou ...]

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For a Borel-measurable function f on \mathbb{R} and Borel-measurable set $E \subset \mathbb{R}$, the *integral of f over E* is

$$\int_E f = \int_{\mathbb{R}} \text{ch}_E \cdot f$$

where ch_E is the characteristic function of E .

5. Convergence theorems: monotone, dominated

Easy, natural examples show that *pointwise* limits $f = \lim_n f_n$ of measurable functions f_n , while still measurable, need *not* satisfy $\int f = \lim \int f_n$. That is, this failure is not a pathology, but, rather, is completely reasonable. Hence additional conditions are essential to know that the integral of a pointwise limit is the limit of the integrals.

First, a relatively simple initial step:

[5.1] Theorem: (*Fatou's lemma*) For Borel-measurable f_n with values in $[0, +\infty]$, the pointwise $f(x) = \liminf_n f_n(x)$ is Borel-measurable, and

$$\int \liminf_n f_n(x) dx \leq \liminf_n \int f_n$$

Proof: [... iou ...] ///

[5.2] Theorem: (*Lebesgue: monotone convergence*) Let f_1, f_2, \dots be a sequence of non-negative real-valued Lebesgue-measurable functions on $[a, b]$, with $f_1(x) \leq f_2(x) \leq \dots$ for all x . Then $\int_a^b \lim_n f_n(x) dx = \lim_n \int_a^b f_n(x) dx$. This includes the possibility that some of the limits of the pointwise values are $+\infty$, and that the integral of the limit is $+\infty$.

Proof: [... iou ...] ///

[5.3] Theorem: (*Lebesgue: dominated convergence*) Let f_1, f_2, \dots be a sequence of complex-valued Lebesgue-measurable functions on $[a, b]$, with $|f_n(x)| \leq g(x)$ for all x , for some measurable g with $\int_a^b g(x) dx < +\infty$. Then $\int_a^b \lim_n f_n(x) dx = \lim_n \int_a^b f_n(x) dx$.

Proof: [... iou ...] ///

6. Urysohn's lemma

Urysohn's lemma proves existence of sufficiently many functions on reasonable topological spaces.

[6.1] Theorem: (*Urysohn*) In a locally compact Hausdorff topological space X , given a compact subset K contained in an open set U , there is a continuous function $0 \leq f \leq 1$ which is 1 on K and 0 off U .

Proof: First, we prove that there is an open set V such that

$$K \subset V \subset \bar{V} \subset U$$

For each $x \in K$ let V_x be an open neighborhood of x with compact closure. By compactness of K , some finite subcollection V_{x_1}, \dots, V_{x_n} of these V_x cover K , so K is contained in the open set $W = \bigcup_i V_{x_i}$ which has compact closure $\bigcup_i \bar{V}_{x_i}$ since the union is *finite*.

Using the compactness again in a similar fashion, for each x in the closed set $X - U$ there is an open W_x containing K and a neighborhood U_x of x such that $W_x \cap U_x = \phi$.

Then

$$\bigcap_{x \in X-U} (X - U) \cap \bar{W} \cap \bar{W}_x = \phi$$

These are compact subsets in a Hausdorff space, so (again from compactness) some *finite* subcollection has empty intersection, say

$$(X - U) \cap (\bar{W} \cap \bar{W}_{x_1} \cap \dots \cap \bar{W}_{x_n}) = \phi$$

That is,

$$\overline{W} \cap \overline{W}_{x_1} \cap \dots \cap \overline{W}_{x_n} \subset U$$

Thus, the open set

$$V = W \cap W_{x_1} \cap \dots \cap W_{x_n}$$

meets the requirements.

Using the possibility of inserting an open subset and its closure between any $K \subset U$ with K compact and U open, we inductively create opens V_r (with compact closures) indexed by rational numbers r in the interval $0 \leq r \leq 1$ such that, for $r > s$,

$$K \subset V_r \subset \overline{V}_r \subset V_s \subset \overline{V}_s \subset U$$

From any such configuration of opens we construct the desired continuous function f by

$$f(x) = \sup\{r \text{ rational in } [0, 1] : x \in V_r, \} = \inf\{r \text{ rational in } [0, 1] : x \in \overline{V}_r, \}$$

It is not immediate that this sup and inf are the same, but if we *grant* their equality then we can prove the *continuity* of this function $f(x)$. Indeed, the sup description expresses f as the supremum of characteristic functions of open sets, so f is at least *lower semi-continuous*.^[1] The inf description expresses f as an infimum of characteristic functions of closed sets so is *upper semi-continuous*. Thus, f would be continuous.

To finish the argument, we must construct the sets V_r and prove equality of the inf and sup descriptions of the function f .

To construct the sets V_i , start by finding V_0 and V_1 such that

$$K \subset V_1 \subset \overline{V}_1 \subset V_0 \subset \overline{V}_0 \subset U$$

Fix a well-ordering r_1, r_2, \dots of the rationals in the open interval $(0, 1)$. Supposing that V_{r_1}, \dots, V_{r_n} have been chosen. let i, j be indices in the range $1, \dots, n$ such that

$$r_j > r_{n+1} > r_i$$

and r_j is the *smallest* among r_1, \dots, r_n above r_{n+1} , while r_i is the *largest* among r_1, \dots, r_n below r_{n+1} . Using the first observation of this argument, find $V_{r_{n+1}}$ such that

$$V_{r_j} \subset \overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_i} \subset \overline{V}_{r_i}$$

This constructs the nested family of opens.

Let $f(x)$ be the sup and $g(x)$ the inf of the characteristic functions above. If $f(x) > g(x)$ then there are $r > s$ such that $x \in V_r$ and $x \notin \overline{V}_s$. But $r > s$ implies that $V_r \subset \overline{V}_s$, so this cannot happen. If $g(x) > f(x)$, then there are rationals $r > s$ such that

$$g(x) > r > s > f(x)$$

Then $s > f(x)$ implies that $x \notin V_s$, and $r < g(x)$ implies $x \in \overline{V}_r$. But $V_r \subset \overline{V}_s$, contradiction. Thus, $f(x) = g(x)$. ///

7. Comparison to continuous functions: Lusin's theorem

[1] A (real-valued) function f is *lower semi-continuous* when for all bounds B the set $\{x : f(x) > B\}$ is open. The function f is *upper semi-continuous* when for all bounds B the set $\{x : f(x) < B\}$ is open. It is easy to show that a sup of lower semi-continuous functions is lower semi-continuous, and an inf of upper semi-continuous functions is upper semi-continuous. As expected, a function both upper and lower semi-continuous is continuous.

One aspect of the following theorem is that we have not inadvertently needlessly included functions wildly unrelated to continuous functions:

[7.1] **Theorem:** (*Lusin*) Continuous functions approximate Borel-measurable functions well: given Borel-measurable real-valued or complex-valued f on \mathbb{R} , for every $\varepsilon > 0$ and for every Borel subset $\Omega \subset \mathbb{R}$ of finite Lebesgue measure, there is a relative closed $E \subset \Omega$ such that $\mu(\Omega - E) < \varepsilon$, and $f|_E$ is *continuous*.

Proof: [... iou ...]

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Not much better can be done than Lusin's theorem says: for example, continuous approximations to the Heaviside step function

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

have to go from 0 to 1 *somewhere*, by the Intermediate Value Theorem, so will be in $(\frac{1}{4}, \frac{3}{4})$ on an open set of strictly positive measure.

[7.2] **Remark:** It turns out that the everyday use of measure theory, measurable functions, and so on, does *not* proceed by way of Lusin's theorem or similar direct connections with continuous functions, but, rather, by direct interaction with the more general ideas.

8. Comparison to uniform pointwise convergence: Severini-Egoroff

[8.1] **Theorem:** (*Severini, Egoroff*) Pointwise convergence of sequences of Borel-measurable functions is approximately *uniform* convergence: given a almost-everywhere pointwise-convergent sequence $\{f_n\}$ of Borel-measurable functions on \mathbb{R} , for every $\varepsilon > 0$ and for every Borel subset $\Omega \subset \mathbb{R}$ of finite Lebesgue measure, there is a Borel subset $E \subset \Omega$ such that $\{f_n\}$ converges *uniformly* pointwise on E .

Proof: [... iou ...]

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[8.2] **Remark:** Despite the connection that the Severini-Egoroff theorem makes between pointwise and *uniform* pointwise convergence, this idea turns out *not* to be the way to understand convergence of measurable functions. Instead, the game becomes ascertaining additional conditions that guarantee convergence of integrals, as earlier.

9. Abstract integration on measure spaces

An elementary but fundamental result is

[9.1] **Proposition:** Let f be a $[0, +\infty]$ -valued measurable function on X . Then there are simple functions s_1, s_2, s_3, \dots with *non-negative real coefficients* so that for all $x \in X$, $s_1(x) \leq s_2(x) \leq s_3(x) \leq \dots \leq f(x)$, and for all $x \in X$, $\lim_n s_n(x) = f(x)$.

Note: Some authors distinguish between *positive* measures and *complex* measures, where the distinction is meant to be that the former are $[0, \infty]$ -valued, while the latter are constrained to assume only '*finite*' complex values.

The *integral of a characteristic function* χ_E is taken to be simply

$$\int_X \chi_E d\mu = \mu(E)$$

Then the *integral of a simple function*

$$s(x) = \sum_{1 \leq i \leq n} c_i \chi_{E_i}$$

(with $c_i \geq 0$) is defined to be

$$\int_X \sum_{1 \leq i \leq n} c_i \chi_{E_i} = \sum_{1 \leq i \leq n} c_i \int_X \chi_{E_i} d\mu = \sum_{1 \leq i \leq n} c_i \int_X \mu E_i$$

For a $[0, +\infty]$ -valued function f , we write

$$0 \leq s \leq f$$

for a *simple function* s if s has *non-negative real coefficients*, and if for all $x \in X$

$$0 \leq s(x) \leq f(x)$$

Then the *Lebesgue integral* of f is defined to be

$$\int_X f d\mu = \sup_{s: 0 \leq s \leq f} \int_X s d\mu$$

Note that at this point we can only integrate *non-negative real-valued functions*.

The standard space

$$L^1(X, \mu) = \{\text{complex-valued measurable } f \text{ so that } \int_X |f| d\mu < \infty\}$$

Since $|f|$ is non-negative real-valued, we can indeed make sense of this. This is the collection of *integrable functions* f . Then write

$$f(x) = u(x) + iv(x)$$

where both u, v are real-valued, and write

$$u = u_+ - u_- \quad v = v_+ - v_-$$

where u_+, v_+ are the ‘positive parts’ and where u_-, v_- are the ‘negative parts’ of these functions. Define the *Lebesgue integral*

$$\int_X f d\mu = \int_X u_+ d\mu - \int_X u_- d\mu + i \int_X v_+ d\mu - i \int_X v_- d\mu$$

Then we have to check that this definition, in terms of integrals of non-negative functions, really has the presumed properties. It is in proving such that we need the *integrability*.

For brevity, when there is no chance of confusion we will often simply write

$$\int_X f$$

rather than either of

$$\int_X f d\mu, \quad \int_X f(x) d\mu(x)$$

for the integral of f on the measure space X with respect to the measure μ .

10. Lebesgue-Radon-Nikodym theorem

Let μ, ν be two positive measures on a common sigma algebra \mathcal{A} on a set X . Say that ν is *absolutely continuous* with respect to μ if $\mu(E) = 0$ implies $\nu(E) = 0$ for all measurable sets E . This is often written $\nu \ll \mu$. The measure μ is *supported on* or *concentrated on* a subset X_o of X if, for all measurable E ,

$$\mu(E) = \mu(E \cap X_o)$$

The two measures μ, ν are *mutually singular* if μ is supported on X_1 and ν is supported on X_2 and $X_1 \cap X_2 = \emptyset$. This is often written $\mu \perp \nu$.

[10.1] Theorem: Theorem. Let μ, ν be positive measures on a common sigma-algebra \mathcal{A} on a set X . There is a unique pair of positive measures ν_a and ν_s so that

$$\nu_a \ll \mu \quad \nu_s \perp \mu$$

Further, there is $\varphi \in L^1(X, \mu)$ so that for any measurable set E

$$\nu_a(E) = \int_X \varphi d\mu$$

The function φ is the *Radon-Nikodym derivative* of ν_a with respect to μ , and is often written as

$$\varphi = \frac{d\nu_a}{d\mu}$$

The pair (ν_a, ν_s) is the *Lebesgue decomposition* of ν with respect to μ .
