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03. Riesz-Markov-Kakutani theorem, Lebesgue measure

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1. Riesz-Markov-Kakutani theorem and regularity
2. Lebesgue measure

1. Riesz-Markov-Kakutani theorem and regularity

Let X be a locally compact, Hausdorff, topological space. A map $f \rightarrow \lambda(f)$ of continuous, compactly supported functions $C_c^o(X)$ to scalars is *positive* when $\lambda(f) \geq 0$ for $f \in C_c^o(X)$ taking values in $[0, +\infty)$.

[1.1] **Theorem:** (*Riesz, Markov, Kakutani, independently*) Given a positive functional λ on $C_c^o(X)$, there is a σ -algebra A containing all Borel sets, and a positive measure μ on A , such that

$$\lambda(f) = \int_X f(x) d\mu(x) \quad (\text{for all } f \in C_c^o(X))$$

- *Outer regularity* holds unconditionally, namely, that for $E \in A$, $\mu(E) = \inf_{U \supset E} \mu(U)$ where U ranges over open sets containing E .
- *Inner regularity* is conditional: for open E , and for $\mu(E) < \infty$, $\mu(E) = \sup_{K \subset E} \mu(K)$ where K ranges over compact sets contained in E .
- μ is *complete*, in the sense that $E' \subset E \in A$ and $\mu(E) = 0$ implies that $E' \in A$.

Proof: (Standard... [... iou ...])

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With a further mild assumption on the physical space X , including familiar spaces such as \mathbb{R}^n , in fact we have unconditional *regularity*:

[1.2] **Theorem:** Suppose further that X is σ -compact, meaning that it is a countable union of compact subsets. Then, in the situation of the previous theorem, μ is unconditionally *inner regular*: $\mu(E) = \sup_{K \subset E} \mu(K)$ as K ranges over compacts contained in E . Thus, the measure μ is a *positive, regular, Borel measure*.

Proof: (Standard... [... iou ...])

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2. Lebesgue measure

As a corollary of the Riesz-Markov-Kakutani theorem we have a different description of the Lebesgue measure and integral, as an extension of the Riemann integral, with the very useful side effect of proving inner and outer regularity.

In the Riesz-Markov-Kakutani theorem, take $X = \mathbb{R}^n$, and $\lambda(f)$ to be the usual Riemann integral for $f \in C_c^o(\mathbb{R}^n)$, and let Lebesgue measure be the associated *positive, regular, Borel* measure. With this description of Lebesgue measure, as opposed to the more tangible (but also more awkward) Lebesgue outer measure, we must verify that all the expected properties do hold.

[2.1] **Corollary:** Let μ be Lebesgue measure, induced by the Riesz-Markov-Kakutani theorem from the Riemann integral on $C_c^0(\mathbb{R}^n)$.

- μ is *translation-invariant* in the sense that $\mu(E + x) = \mu(E)$ for all $x \in \mathbb{R}^n$.
- The Lebesgue measure of a cube $(a_1, b_1) \times \dots \times (a_n, b_n)$ is the product $\prod_i |b_i - a_i|$, and similarly for closed and half-open intervals and their products.

Proof: (Standard... [... iou ...])

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