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Introduction to Levi-Sobolev spaces

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2017-18/11_intro_Sobolev.pdf]

The simplest case of a Levi-Sobolev *imbedding theorem* asserts that the +1-index Levi-Sobolev Hilbert space $H^1[a, b]$ described below is inside $C^o[a, b]$. This is a corollary of a Levi-Sobolev *inequality* asserting that the $C^o[a, b]$ norm is *dominated* by the $H^1[a, b]$ norm. All that is used is the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakovsky inequality. The point is that there is a large *Hilbert space* $H^1[a, b]$ inside the *Banach space* $C^o[a, b]$.

We will do much more with this idea subsequently.

We can think of $L^2[a, b]$ as

$$L^2[a, b] = \text{completion of } C^o[a, b] \text{ with respect to } \|f\|_{L^2} = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}$$

In fact, by this point we have shown that every $C^k[a, b]$ is dense in $L^2[a, b]$.

The +1-index *Levi-Sobolev space*^[1] $H^1[a, b]$ is

$$H^1[a, b] = \text{completion of } C^1[a, b] \text{ with respect to } \|f\|_{H^1} = \left(\|f\|_{L^2[a, b]}^2 + \|f'\|_{L^2[a, b]}^2 \right)^{1/2}$$

[0.1] Theorem: (*Levi-Sobolev inequality*) On $C^1[a, b]$, the $H^1[a, b]$ -norm *dominates* the $C^o[a, b]$ -norm. That is, there is a constant C depending only on a, b such that $\|f\|_{C^o[a, b]} \leq C \cdot \|f\|_{H^1[a, b]}$ for every $f \in C^1[a, b]$.

Proof: For $a \leq x \leq y \leq b$, for $f \in C^1[a, b]$, the fundamental theorem of calculus gives

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_x^y f'(t) dt \right| \leq \int_x^y |f'(t)| dt \leq \left(\int_x^y |f'(t)|^2 dt \right)^{1/2} \cdot \left(\int_x^y 1 dt \right)^{1/2} \\ &\leq \|f'\|_{L^2} \cdot |x - y|^{1/2} \leq \|f'\|_{L^2} \cdot |a - b|^{1/2} \end{aligned}$$

Using the continuity of $f \in C^1[a, b]$, let $y \in [a, b]$ be such that $|f(y)| = \min_x |f(x)|$. Using the previous inequality,

$$\begin{aligned} |f(x)| &\leq |f(y)| + |f(x) - f(y)| \leq \frac{\int_a^b |f(t)| dt}{|a - b|} + |f(x) - f(y)| \leq \frac{\int_a^b |f| \cdot 1}{|a - b|} + \|f'\|_{L^2} \cdot |a - b|^{1/2} \\ &\leq \frac{\|f\|_{L^2}^{1/2} \cdot |a - b|^{1/2}}{|a - b|} + \|f'\|_{L^2} \cdot |a - b|^{1/2} = \frac{\|f\|_{L^2}^{1/2}}{|a - b|^{1/2}} + \|f'\|_{L^2} \cdot |a - b|^{1/2} \leq \left(\|f\|_{L^2} + \|f'\|_{L^2} \right) \cdot \left(|a - b|^{-1/2} + |a - b|^{1/2} \right) \\ &\leq 2(\|f\|^2 + \|f'\|^2)^{1/2} \cdot \left(|a - b|^{-1/2} + |a - b|^{1/2} \right) = \|f\|_{H^1} \cdot 2 \left(|a - b|^{-1/2} + |a - b|^{1/2} \right) \end{aligned}$$

Thus, on $C^1[a, b]$ the H^1 norm dominates the C^o -norm. ///

[0.2] Corollary: (*Levi-Sobolev imbedding*) $H^1[a, b] \subset C^o[a, b]$.

[1] ... also denoted $W^{1,2}[a, b]$, where the superscript 2 refers to L^2 , rather than L^p . Beppo Levi noted the importance of taking Hilbert space completion with respect to this norm in 1906, giving a correct formulation of *Dirichlet's principle*. Sobolev's systematic development of these ideas was in the mid-1930's.

Proof: Since $H^1[a, b]$ is the H^1 -norm completion of $C^1[a, b]$, every $f \in H^1[a, b]$ is an H^1 -limit of functions $f_n \in C^1[a, b]$. That is, $\|f - f_n\|_{H^1[a, b]} \rightarrow 0$. Since the H^1 -norm dominates the C^0 -norm, $\|f - f_n\|_{C^0[a, b]} \rightarrow 0$. A C^0 limit of continuous functions is continuous, so f is continuous. ///

In fact, we have a stronger conclusion than continuity, namely, a *Lipschitz condition* with exponent $\frac{1}{2}$:

[0.3] **Corollary:** (of proof of theorem) $|f(x) - f(y)| \leq \|f'\|_{L^2} \cdot |x - y|^{\frac{1}{2}}$ for $f \in H^1[a, b]$. ///
