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15a. Operators on Hilbert spaces

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1. Boundedness, continuity, operator norms

A linear (not necessarily continuous) map $T : X \rightarrow Y$ from one Hilbert space to another is *bounded* if, for all $\varepsilon > 0$, there is $\delta > 0$ such that $|Tx|_Y < \varepsilon$ for all $x \in X$ with $|x|_X < \delta$.

[1.1] **Proposition:** For a linear, not necessarily continuous, map $T : X \rightarrow Y$ of Hilbert spaces, the following three conditions are equivalent:

- (i) T is continuous
- (ii) T is continuous at 0
- (iii) T is bounded

Proof: For T continuous at 0, given $\varepsilon > 0$ and $x \in X$, there is small enough $\delta > 0$ such that $|Tx' - 0|_Y < \varepsilon$ for $|x' - 0|_X < \delta$. For $|x'' - x|_X < \delta$, using the linearity,

$$|Tx'' - Tx|_Y = |T(x'' - x)|_Y < \varepsilon$$

That is, continuity at 0 implies continuity.

Since $|x| = |x - 0|$, continuity at 0 is immediately equivalent to boundedness. ///

[1.2] **Definition:** The *kernel* $\ker T$ of a linear (not necessarily continuous) linear map $T : X \rightarrow Y$ from one Hilbert space to another is

$$\ker T = \{x \in X : Tx = 0 \in Y\}$$

[1.3] **Proposition:** The kernel of a continuous linear map $T : X \rightarrow Y$ is closed.

Proof: For T continuous

$$\ker T = T^{-1}\{0\} = X - T^{-1}(Y - \{0\}) = X - T^{-1}(\text{open}) = X - \text{open} = \text{closed}$$

since the inverse images of open sets by a continuous map are open. ///

[1.4] **Definition:** The *operator norm* $|T|$ of a linear map $T : X \rightarrow Y$ is

$$\text{operator norm } T = |T| = \sup_{x \in X : |x|_X \leq 1} |Tx|_Y$$

[1.5] **Corollary:** A linear map $T : X \rightarrow Y$ is continuous if and only if its operator norm is finite. ///

2. Adjoint

An *adjoint* T^* of a continuous linear map $T : X \rightarrow Y$ from a pre-Hilbert space X to a pre-Hilbert space Y (if T^* exists) is a continuous linear map $T^* : Y \rightarrow X$ such that

$$\langle Tx, y \rangle_Y = \langle x, T^*y \rangle_X$$

[2.1] **Remark:** When a pre-Hilbert space X is not complete, that is, is not a Hilbert space, an operator $T : X \rightarrow Y$ may fail to have an adjoint.

[2.2] **Theorem:** A continuous linear map $T : X \rightarrow Y$ from a *Hilbert* space X to a Hilbert space Y has a unique adjoint T^* .

[2.3] **Remark:** In fact, the target space of T need not be a Hilbert space, that is, need not be complete, but we will not use this.

Proof: For each $y \in Y$, the map

$$\lambda_y : X \rightarrow \mathbb{C}$$

given by

$$\lambda_y(x) = \langle Tx, y \rangle$$

is a continuous linear functional on X . By Riesz-Fréchet, there is a unique $x_y \in X$ so that

$$\langle Tx, y \rangle = \lambda_y(x) = \langle x, x_y \rangle$$

Try to define T^* by $T^*y = x_y$. This is a well-defined map from Y to X , and has the adjoint property $\langle Tx, y \rangle_Y = \langle x, T^*y \rangle_X$.

To prove that T^* is continuous, prove that it is bounded. From Cauchy-Schwarz-Bunyakovsky

$$|T^*y|^2 = |\langle T^*y, T^*y \rangle_X| = |\langle y, TT^*y \rangle_Y| \leq |y| \cdot |TT^*y| \leq |y| \cdot |T| \cdot |T^*y|$$

where $|T|$ is the operator norm. For $T^*y \neq 0$, divide by it to find

$$|T^*y| \leq |y| \cdot |T|$$

Thus, $|T^*| \leq |T|$. In particular, T^* is bounded. Since $(T^*)^* = T$, by symmetry $|T| = |T^*|$. Linearity of T^* is easy. ///

[2.4] **Corollary:** For a continuous linear map $T : X \rightarrow Y$ of Hilbert spaces, $T^{**} = T$. ///

An operator $T \in \text{End}(X)$ commuting with its adjoint is *normal*, that is,

$$TT^* = T^*T$$

This only applies to operators from a Hilbert space *to itself*. An operator T is *self-adjoint* or *hermitian* if $T = T^*$. That is, T is hermitian when

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad (\text{for all } x, y \in X)$$

An operator T is *unitary* when

$$TT^* = \text{identity map on } Y \quad T^*T = \text{identity map on } X$$

There are simple examples in infinite-dimensional spaces where $TT^* = 1$ does not imply $T^*T = 1$, and vice-versa. Thus, it does *not* suffice to check something like $\langle Tx, Tx \rangle = \langle x, x \rangle$ to prove unitariness. Obviously hermitian operators are normal, as are unitary operators, using the more careful definition.

3. Stable subspaces and complements

Let $T : X \rightarrow X$ be a continuous linear operator on a Hilbert space X . A vector subspace Y is *T-stable* or *T-invariant* if $Ty \in Y$ for all $y \in Y$. Often one is most interested in the case that the subspace be *closed* in addition to being *invariant*.

[3.1] Proposition: For $T : X \rightarrow X$ a continuous linear operator on a Hilbert space X , and Y a T -stable subspace, Y^\perp is T^* -stable.

Proof: For $z \in Y^\perp$ and $y \in Y$,

$$\langle T^*z, y \rangle = \langle z, T^{**}y \rangle = \langle z, Ty \rangle$$

since $T^{**} = T$, from above. Since Y is T -stable, $Ty \in Y$, and this inner product is 0, and $T^*z \in Y^\perp$.

///

[3.2] Corollary: For continuous *self-adjoint* T on a Hilbert space X , and Y a T -stable subspace, both Y and Y^\perp are T -stable. ///

[3.3] Remark: *Normality* of $T : X \rightarrow X$ is insufficient to assure the conclusion of the corollary, in general. For example, with the two-sided ℓ^2 space

$$X = \{ \{c_n : n \in \mathbb{Z}\} : \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty \}$$

the right-shift operator

$$(Tc)_n = c_{n-1} \quad (\text{for } n \in \mathbb{Z})$$

has adjoint the *left* shift operator

$$(T^*c)_n = c_{n+1} \quad (\text{for } n \in \mathbb{Z})$$

and

$$T^*T = TT^* = 1_X$$

So this T is not merely *normal*, but *unitary*. However, the T -stable subspace

$$Y = \{ \{c_n\} \in X : c_k = 0 \text{ for } k < 0 \}$$

is not T^* -stable, nor is its orthogonal complement T -stable.

On the other hand, adjoint-stable *collections* of operators have a good stability result:

[3.4] Proposition: Suppose for every T in a set A of bounded linear operators on a Hilbert space V the adjoint T^* is also in A . Then, for an A -stable subspace W of V , the orthogonal complement W^\perp is also A -stable.

Proof: For y in W^\perp and $T \in A$, for $x \in W$,

$$\langle x, Ty \rangle = \langle T^*x, y \rangle \in \langle W, y \rangle = \{0\}$$

since $T^* \in A$. ///

4. Spectrum, eigenvalues

For a continuous linear operator $T \in \text{End}(X)$, the λ -eigenspace of T is

$$X_\lambda = \{x \in X : Tx = \lambda x\}$$

If this space is not simply $\{0\}$, then λ is an *eigenvalue*.

[4.1] Proposition: An eigenspace X_λ for a continuous linear operator T on X is a *closed* and T -stable subspace of X . For *normal* T the adjoint T^* acts by the scalar $\bar{\lambda}$ on X_λ .

Proof: The λ -eigenspace is the kernel of the continuous linear map $T - \lambda$, so is closed. The stability is clear, since the operator restricted to the eigenspace is a scalar operator. For $v \in X_\lambda$, using normality,

$$(T - \lambda)T^*v = T^*(T - \lambda)v = T^* \cdot 0 = 0$$

Thus, X_λ is T^* -stable. For $x, y \in X_\lambda$,

$$\langle (T^* - \bar{\lambda})x, y \rangle = \langle x, (T - \lambda)y \rangle = \langle x, 0 \rangle$$

Thus, $(T^* - \bar{\lambda})x = 0$. ///

[4.2] Proposition: For T *normal*, for $\lambda \neq \mu$, and for $x \in X_\lambda, y \in X_\mu$, always $\langle x, y \rangle = 0$. For T *self-adjoint*, if $X_\lambda \neq 0$ then $\lambda \in \mathbb{R}$. For T *unitary*, if $X_\lambda \neq 0$ then $|\lambda| = 1$.

Proof: Let $x \in X_\lambda, y \in X_\mu$, with $\mu \neq \lambda$. Then

$$\lambda \langle x, y \rangle = \langle Tx, y \rangle = \langle x, T^*y \rangle = \langle x, \bar{\mu}y \rangle = \bar{\mu} \langle x, y \rangle$$

invoking the previous result. Thus,

$$(\lambda - \bar{\mu}) \langle x, y \rangle = 0$$

giving the result. For T self-adjoint and x a non-zero λ -eigenvector,

$$\lambda \langle x, x \rangle = \langle x, T^*x \rangle = \langle x, Tx \rangle = \langle x, \lambda x \rangle = \bar{\lambda} \langle x, x \rangle$$

Thus, $(\lambda - \bar{\lambda}) \langle x, x \rangle = 0$. Since x is non-zero, the result follows. For T unitary and x a non-zero λ -eigenvector,

$$\langle x, x \rangle = \langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = |\lambda|^2 \cdot \langle x, x \rangle$$

///

In what follows, for a complex scalar λ write simply λ for scalar multiplication by λ on a Hilbert space X .

[4.3] Definition: The *spectrum* $\sigma(T)$ of a continuous linear operator $T : X \rightarrow X$ on a Hilbert space X is the collection of complex numbers λ such that $T - \lambda$ *does not have* a continuous linear inverse.

[4.4] Definition: The *discrete spectrum* $\sigma_{\text{disc}}(T)$ is the collection of complex numbers λ such that $T - \lambda$ fails to be *injective*. In other words, the discrete spectrum is the collection of eigenvalues.

[4.5] Definition: The *continuous spectrum* $\sigma_{\text{cont}}(T)$ is the collection of complex numbers λ such that $T - \lambda \cdot 1_X$ *is* injective, *does* have dense image, but fails to be *surjective*.

[4.6] Definition: The *residual spectrum* $\sigma_{\text{res}}(T)$ is everything else: neither discrete nor continuous spectrum. That is, the residual spectrum of T is the collection of complex numbers λ such that $T - \lambda \cdot 1_X$ *is* injective, and *fails* to have dense image (so is certainly not surjective).

[4.7] **Remark:** To see that there are no *other* possibilities, note that the *Closed Graph Theorem* implies that a bijective, continuous, linear map $T : X \rightarrow Y$ of Banach spaces has continuous inverse. Indeed, granting that the inverse exists as a linear map, its graph is

$$\text{graph of } T^{-1} = \{(y, x) \in Y \times X : (x, y) \text{ in the graph of } T \subset X \times Y\}$$

Since the graph of T is closed, the graph of T^{-1} is closed, and by the Closed Graph Theorem T^{-1} is continuous.

The potential confusion of *residual* spectrum does not occur in many situations of interest”

[4.8] **Proposition:** A normal operator $T : X \rightarrow X$ has *empty* residual spectrum.

Proof: The adjoint of $T - \lambda$ is $T^* - \bar{\lambda}$, so consider $\lambda = 0$ to lighten the notation. Suppose that T does *not* have dense image. Then there is non-zero z such that

$$0 = \langle z, Tx \rangle = \langle T^*z, x \rangle \quad (\text{for every } x \in X)$$

Therefore $T^*z = 0$, and the 0-eigenspace Z of T^* is non-zero. Since $T^*(Tz) = T(T^*z) = T(0) = 0$ for $z \in Z$, T^* stabilizes Z . That is, Z is both T and T^* -stable. Therefore, $T = (T^*)^*$ acts on Z by (the complex conjugate of) 0, and T has non-trivial 0-eigenvectors, contradiction. ///

5. Generalities on spectra

It is convenient to know that spectra of continuous operators are *non-empty, compact* subsets of \mathbb{C} .

Knowing this, *every* non-empty compact subset of \mathbb{C} is easily made to appear as the spectrum of a continuous operator, even *normal* ones, as below.

[5.1] **Proposition:** The spectrum $\sigma(T)$ of a continuous linear operator $T : V \rightarrow V$ on a Hilbert space V is *bounded* by the operator norm $|T|_{\text{op}}$.

Proof: For $|\lambda| > |T|_{\text{op}}$, an obvious heuristic suggests an expression for the *resolvent* $R_\lambda = (T - \lambda)^{-1}$:

$$(T - \lambda)^{-1} = -\lambda^{-1} \cdot \left(1 - \frac{T}{\lambda}\right)^{-1} = -\lambda^{-1} \cdot \left(1 + \frac{T}{\lambda} + \left(\frac{T}{\lambda}\right)^2 + \dots\right)$$

The infinite series converges in operator norm for $|T/\lambda|_{\text{op}} < 1$, that is, for $|\lambda| > |T|_{\text{op}}$. Then

$$(T - \lambda) \cdot (-\lambda^{-1}) \cdot \left(1 + \frac{T}{\lambda} + \left(\frac{T}{\lambda}\right)^2 + \dots\right) = 1$$

giving a continuous inverse $(T - \lambda)^{-1}$, so $\lambda \notin \sigma(T)$. ///

[5.2] **Remark:** The same argument applied to T^n shows that $\sigma(T^n)$ is inside the closed ball of radius $|T^n|_{\text{op}}$. By the elementary identity

$$T^n - \lambda^n = (T - \lambda) \cdot (T^{n-1} + T^{n-2}\lambda + \dots + T\lambda^{n-2} + \lambda^{n-2})$$

$(T - \lambda)^{-1}$ exists for $|\lambda^n| > |T^n|_{\text{op}}$, that is, for $|\lambda| > |T^n|_{\text{op}}^{1/n}$. That is, $\sigma(T)$ is inside the closed ball of radius $\inf_{n \geq 1} |T^n|_{\text{op}}^{1/n}$. The latter expression is the *spectral radius* of T . This notion is relevant to *non-normal* operators, such as the *Volterra operator*, whose spectral radius is 0, while its operator norm is much larger.

[5.3] **Proposition:** The spectrum $\sigma(T)$ of a continuous linear operator $T : V \rightarrow V$ on a Hilbert space V is *compact*.

Proof: That $\lambda \notin \sigma(T)$ is that there is a continuous linear operator $(T - \lambda)^{-1}$. We claim that for μ sufficiently close to λ , $(T - \mu)^{-1}$ exists. Indeed, a heuristic suggests an expression for $(T - \mu)^{-1}$ in terms of $(T - \lambda)^{-1}$. The algebra is helpfully simplified by replacing T by $T + \lambda$, so that $\lambda = 0$. With μ near 0 and granting existence of T^{-1} , the heuristic is

$$(T - \mu)^{-1} = (1 - \mu T^{-1})^{-1} \cdot T^{-1} = \left(1 + \mu T^{-1} + (\mu T^{-1})^2 + \dots\right) \cdot T^{-1}$$

The geometric series converges in operator norm for $|\mu T^{-1}|_{\text{op}} < 1$, that is, for $|\mu| < |T^{-1}|_{\text{op}}^{-1}$. Having found the obvious candidate for an inverse,

$$(1 - \mu T^{-1}) \cdot \left(1 + \mu T^{-1} + (\mu T^{-1})^2 + \dots\right) = 1$$

and

$$(T - \mu) \cdot \left(1 + \mu T^{-1} + (\mu T^{-1})^2 + \dots\right) \cdot T^{-1} = 1$$

so there is a continuous linear operator $(T - \mu)^{-1}$, and $\mu \notin \sigma(T)$. Having already proven that $\sigma(T)$ is *bounded*, it is *compact*. ///

[5.4] Proposition: The spectrum $\sigma(T)$ of a continuous linear operator on a Hilbert space $V \neq \{0\}$ is *non-empty*.

Proof: The argument reduces the issue to Liouville's theorem from complex analysis, that a *bounded* entire (holomorphic) function is *constant*. Further, an entire function that goes to 0 at ∞ is identically 0.

Suppose the resolvent $R_\lambda = (T - \lambda)^{-1}$ is a continuous linear operator for all $\lambda \in \mathbb{C}$. The operator norm is readily estimated for large λ :

$$\begin{aligned} |R_\lambda|_{\text{op}} &= |\lambda|^{-1} \cdot \left|1 + \frac{T}{\lambda} + \left(\frac{T}{\lambda}\right)^2 + \dots\right|_{\text{op}} \\ &\leq |\lambda|^{-1} \cdot \left(1 + \left|\frac{T}{\lambda}\right|_{\text{op}} + \left|\frac{T}{\lambda}\right|_{\text{op}}^2 + \dots\right) = \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{|T|_{\text{op}}}{|\lambda|}} \end{aligned}$$

This goes to 0 as $|\lambda| \rightarrow \infty$. *Hilbert's identity* asserts the complex differentiability as operator-valued function:

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} = R_\lambda \cdot \frac{(T - \mu) - (T - \lambda)}{\lambda - \mu} \cdot R_\mu = R_\lambda \cdot R_\mu \longrightarrow R_\lambda^2 \quad (\text{as } \mu \rightarrow \lambda)$$

since $\mu \rightarrow R_\mu$ is continuous for large μ , by the same identity:

$$|R_\lambda - R_\mu|_{\text{op}} \leq |\lambda - \mu| \cdot |R_\mu \cdot R_\lambda|_{\text{op}}$$

Thus, the scalar-valued functions $\lambda \rightarrow \langle R_\lambda v, w \rangle$ for $v, w \in V$ are complex-differentiable, and satisfy

$$|\langle R_\lambda v, w \rangle| \leq |R_\lambda v| \cdot |w| \leq |R_\lambda|_{\text{op}} \cdot |v| \cdot |w| \leq \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{|T|_{\text{op}}}{|\lambda|}} \cdot |v| \cdot |w|$$

By Liouville, $\langle R_\lambda v, w \rangle = 0$ for all $v, w \in V$, which is impossible. Thus, the spectrum is not empty. ///

[5.5] Proposition: The entire spectrum, both point-spectrum and continuous-spectrum, of a *self-adjoint* operator is a non-empty, compact subset of \mathbb{R} . The entire spectrum of a *unitary* operator is a non-empty, compact subset of the unit circle.

Proof: For self-adjoint T , we claim that the imaginary part of $\langle (T - \mu)v, v \rangle$ is at least $\langle v, v \rangle$ times the imaginary part of μ . Indeed, $\langle Tv, v \rangle$ is *real*, since

$$\langle Tv, v \rangle = \langle v, T^*v \rangle = \langle v, Tv \rangle = \overline{\langle Tv, v \rangle}$$

so

$$\langle (T - \mu)v, v \rangle = \langle Tv, v \rangle - \mu \cdot \langle v, v \rangle$$

and

$$|\operatorname{Im} \langle (T - \mu)v, v \rangle| = |\operatorname{Im} \mu| \cdot \langle v, v \rangle$$

and by Cauchy-Schwarz-Bunyakovsky

$$|(T - \mu)v| \cdot |v| \geq |\langle (T - \mu)v, v \rangle| \geq |\operatorname{Im} \mu| \cdot \langle v, v \rangle = |\operatorname{Im} \mu| \cdot |v|^2$$

Dividing by $|v|$,

$$|(T - \mu)v| \geq |\operatorname{Im} \mu| \cdot |v|$$

This inequality shows more than the injectivity of $T - \mu$. Namely, the inequality gives a bound on the operator norm of the inverse $(T - \mu)^{-1}$ defined on the image of $T - \mu$. The image is *dense* since μ is not an eigenvalue and there is no residual spectrum for normal operators T . Thus, the inverse extends by continuity to a continuous linear map defined on the whole Hilbert space. Thus, $T - \mu$ has a continuous linear inverse, and μ is not in the spectrum of T .

For T unitary, $|Tv| = |v|$ for all v implies $T|_{\text{op}} = 1$. Thus, $\sigma(T)$ is contained in the unit disk, by the general bound on spectra in terms of operator norms. From $(T - \lambda)^* = T^* - \bar{\lambda}$, the spectrum of T^* is obtained by complex-conjugating the spectrum of T . Thus, for unitary T , the spectrum of $T^{-1} = T^*$ is also contained in the unit disk. At the same time, the natural

$$T - \lambda = -T \cdot (T^{-1} - \lambda^{-1}) \cdot \lambda$$

gives

$$(T - \lambda)^{-1} = -\lambda^{-1} \cdot (T^{-1} - \lambda^{-1})^{-1} \cdot T^{-1}$$

so $\lambda^{-1} \in \sigma(T^{-1})$ exactly when $\lambda \in \sigma(T)$. Thus, the spectra of both T and $T^* = T^{-1}$ are inside the unit circle. ///

[5.6] Remark:

6. Positive examples

Let ℓ^2 be the usual space of square-summable sequences (a_1, a_2, \dots) , with standard orthonormal basis

$$e_j = \underbrace{(0, \dots, 0, 1, 0, \dots)}_{1 \text{ at } j\text{th position}}$$

[6.1] **Multiplication operators with specified eigenvalues** Given a countable, bounded list of complex numbers λ_j , the operator $T : \ell^2 \rightarrow \ell^2$ by

$$T : (a_1, a_2, \dots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \dots)$$

has λ_j -eigenvector the standard basis element e_j . Clearly

$$T^* : (a_1, a_2, a_3, \dots) \longrightarrow (\bar{\lambda}_1 \cdot a_1, \bar{\lambda}_2 \cdot a_2, \bar{\lambda}_3 \cdot a_3, \dots)$$

so T is *normal*, in the sense that $TT^* = T^*T$. To see that there are no *other* eigenvalues, suppose $Tv = \mu \cdot v$ with μ not among the λ_j . Then

$$\mu \cdot \langle v, e_j \rangle = \langle Tv, e_j \rangle = \langle v, T^*e_j \rangle = \langle v, \bar{\lambda}_j e_j \rangle = \lambda_j \cdot \langle v, e_j \rangle$$

Thus, $(\mu - \lambda_j) \cdot \langle v, e_j \rangle = 0$, and $\langle v, e_j \rangle = 0$ for all j . Since e_j form an orthonormal basis, $v = 0$. ///

[6.2] Every compact subset of \mathbb{C} is the spectrum of an operator Grant for the moment a countable dense subset $\{\lambda_j\}$ of a non-empty compact subset^[1] C of \mathbb{C} , and as above let

$$T : (a_1, a_2, a_3, \dots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \lambda_3 \cdot a_3, \dots)$$

We saw that there are no further eigenvalues. Since spectra are *closed*, the closure C of $\{\lambda_j\}$ is *contained* in $\sigma(T)$.

It remains to show that the continuous spectrum is no larger than the closure C of the eigenvalues, *in this example*. That is, for $\mu \notin C$, exhibit a continuous linear $(T - \mu)^{-1}$.

For $\mu \notin C$, there is a uniform lower bound $0 < \delta \leq |\mu - \lambda_j|$. That is, $\sup_j |\mu - \lambda_j|^{-1} \leq \delta^{-1}$. Thus, the naturally suggested map

$$(a_1, a_2, \dots) \longrightarrow \left((\lambda_1 - \mu)^{-1} \cdot a_1, (\lambda_2 - \mu)^{-1} \cdot a_2, \dots \right)$$

is a bounded linear map, and gives $(T - \mu)^{-1}$.

[6.3] Two-sided shift has no eigenvalues Let V be the Hilbert space of *two-sided* sequences $(\dots, a_{-1}, a_0, a_1, \dots)$ with natural inner product

$$\langle (\dots, a_{-1}, a_0, a_1, \dots), (\dots, b_{-1}, b_0, b_1, \dots) \rangle = \dots + a_{-1}b_{-1} + a_0b_0 + a_1b_1 + \dots$$

The right and left *two-sided* shift operators are

$$(R \cdot a)_n = a_{n-1} \qquad (L \cdot a)_n = a_{n+1}$$

These operators are mutual adjoints, mutual inverses, so are unitary. Being unitary, their operator norms are 1, so their spectra are non-empty compact subsets of the unit circle.

They have no eigenvalues: indeed, for $Rv = \lambda \cdot v$, if there is any index n with $v_n \neq 0$, then the relation $Rv = \lambda \cdot v$ gives $v_{n+k+1} = \lambda \cdot v_{n+k}$ for $k = 0, 1, 2, \dots$. Since $|\lambda| = 1$, such a vector is not in ℓ^2 .

Nevertheless, we claim that $\lambda \in \sigma(L)$ for every λ with $|\lambda| = 1$, and similarly for R . Indeed, for λ *not* in the spectrum, there is a continuous linear operator $(L - \lambda)^{-1}$, so $|(L - \lambda)v| \geq \delta \cdot |v|$ for some $\delta > 0$. It is easy to make *approximate* eigenvectors for L for any $|\lambda| = 1$: let

$$v^{(\ell)} = (\dots, 0, \dots, 0, 1, \lambda, \lambda^2, \lambda^3, \dots, \lambda^\ell, 0, 0, \dots)$$

Obviously it doesn't matter where the non-zero entries begin. From

$$(L - \lambda)v^{(\ell)} = (\dots, 0, \dots, 0, 1, 0, \dots, 0, \lambda^{\ell+1}, 0, 0, \dots)$$

$|(L - \lambda)v^{(\ell)}| = \sqrt{1 + 1}$, while $|v^{(\ell)}| = \sqrt{\ell + 1}$. Thus, $|(L - \lambda)v^{(\ell)}|/|v^{(\ell)}| \longrightarrow 0$, and there can be no $(L - \lambda)^{-1}$. Thus, every λ on the unit circle is in $\sigma(R)$.

[1] To make a countable dense subset of C , for $n = 1, 2, \dots$ cover C by finitely-many disks of radius $1/n$, each meeting C , and in each choose a point of C . The union over $n = 1, 2, \dots$ of these finite sets is countable and dense in C .

[6.4] Compact multiplication operators on ℓ^2 For a sequence of complex numbers $\lambda_n \rightarrow 0$, we claim that the multiplication operator

$$T : (a_1, a_2, \dots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \dots)$$

is *compact*. We already showed that it has eigenvalues exactly $\lambda_1, \lambda_2, \dots$, and spectrum the *closure* of $\{\lambda_j\}$. Thus, the spectrum includes 0, but 0 is an *eigenvalue* only when it appears among the λ_j , which may range from 0 times to infinitely-many times.

To prove that the operator is compact, we prove that the image of the unit ball is pre-compact, by showing that it is *totally bounded*. Given $\varepsilon > 0$, take k such that $|\lambda_i| < \varepsilon$ for $i > k$. The ball in \mathbb{C}^k of radius $\max\{|\lambda_j| : j \leq k\}$ is precompact, so has a finite cover by ε -balls, centered at points v^1, \dots, v^N . For $v = (v_1, v_2, \dots)$ with $|v| \leq 1$,

$$Tv = (\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_k v_k, 0, 0, \dots) + (0, \dots, 0, \lambda_{k+1} v_{k+1}, \lambda_{k+2} v_{k+2}, \dots)$$

With v^j the closest of the v^1, \dots, v^N to $(\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_k v_k, 0, 0, \dots)$,

$$|Tv - v^j| < \varepsilon + |(0, \dots, 0, \lambda_{k+1} v_{k+1}, \lambda_{k+2} v_{k+2}, \dots)| < \varepsilon + \varepsilon \cdot |(0, \dots, 0, v_{k+1}, v_{k+2}, \dots)| \leq \varepsilon + \varepsilon \cdot |v| \leq 2\varepsilon$$

Thus, the image of the unit ball under T is covered by finitely-many 2ε -balls. ///

[6.5] Multiplication operators on $L^2[a, b]$ For $\varphi \in C^o[a, b]$, we claim that the multiplication operator

$$M_\varphi : L^2[a, b] \longrightarrow L^2[a, b]$$

by

$$M_\varphi f(x) = \varphi(x) \cdot f(x)$$

is *normal*, and has spectrum the image $\varphi[a, b]$ of φ . The eigenvalues are λ such that $\varphi(x) = \lambda$ on a subset of $[a, b]$ of positive measure. The normality is clear, so, beyond eigenvalues, we need only examine continuous spectrum, not residual.

On one hand, if $\varphi(x) = \lambda$ on a set of positive measure, there is an infinite-dimensional sub-space of $L^2[0, 1]$ of functions supported there, and all these are eigenvectors. On the other hand, if $f \neq 0$ in $L^2[0, 1]$ and $\varphi(x) \cdot f(x) = \lambda \cdot f(x)$, even if f is altered on a set of measure 0, it must be that $\varphi(x) = \lambda$ on a set of positive measure.

To understand the continuous spectrum, for $\varphi(x_o) = \lambda$ make *approximate eigenvectors* by taking L^2 functions f supported on $[x_o - \delta, x_o + \delta]$, where $\delta > 0$ is small enough so that $|\varphi(x) - \varphi(x_o)| < \varepsilon$ for $|x - x_o| < \delta$. Then

$$\|(M_\varphi - \lambda)f\|_{L^2}^2 = \int |\varphi(x) - \lambda|^2 \cdot |f(x)|^2 dx \leq \varepsilon^2 \cdot \|f\|_{L^2}^2$$

Thus, $\inf_{f \neq 0} \|(M_\varphi - \lambda)f\|_{L^2} / \|f\|_{L^2} = 0$, so $M_\varphi - \lambda$ is not invertible. If λ is not an eigenvalue, it is in the continuous spectrum. On the other hand, if $\varphi(x) \neq \lambda$, then there is some $\delta > 0$ such that $|\varphi(x) - \lambda| \geq \delta$ for all $x \in [0, 1]$, by the compactness of $[0, 1]$. Then

$$\|(M_\varphi - \lambda)f\|_{L^2}^2 = \int_0^1 |\varphi(x) - \lambda|^2 \cdot |f(x)|^2 dx \geq \int_0^1 \delta^2 \cdot |f(x)|^2 dx = \delta^2 \cdot \|f\|_{L^2}^2$$

Thus, there is a continuous inverse $(M_\varphi - \lambda)$, and λ is *not* in the spectrum.

7. Cautionary examples: non-normal operators

[7.1] Shift operators on one-sided ℓ^2 We claim the following: The right-shift

$$R : (a_1, a_2, \dots) \longrightarrow (0, a_1, a_2, \dots)$$

and the left-shift

$$L : (a_1, a_2, a_3, \dots) \longrightarrow (a_2, \dots)$$

are mutual adjoints. These operators are not normal, since $L \circ R = 1_{\ell^2}$ but

$$R \circ L : (a_1, a_2, \dots) \longrightarrow (0, a_2, \dots)$$

The eigenvalues of the left-shift L are all complex numbers in the open unit disk in \mathbb{C} . In particular, there is a *continuum* of eigenvalues and eigenvectors, so they *cannot be mutually orthogonal*. The spectrum $\sigma(L)$ is the closed unit disk.

The right-shift R has *no* eigenvalues, has continuous spectrum the unit circle, and residual spectrum the open unit disk with 0 removed.

Indeed, suppose

$$(0, a_1, a_2, \dots) = R(a_1, a_2, \dots) = \lambda \cdot (a_1, a_2, \dots)$$

With n the lowest index such that $a_n \neq 0$, the n^{th} component in the eigenvector relation gives $0 = a_{n-1} = \lambda \cdot a_n$, so $\lambda = 0$. Then, the $(n+1)^{\text{th}}$ component gives $a_n = \lambda \cdot a_{n+1} = 0$, contradiction. This proves that R has *no* eigenvalues.

Oppositely, for $|\lambda| < 1$,

$$L(1, \lambda, \lambda^2, \dots) = (\lambda, \lambda^2, \dots) = \lambda \cdot (1, \lambda, \lambda^2, \dots)$$

so every such λ is an eigenvector for L . On the other hand, for $|\lambda| = 1$, in an eigenvector relation

$$(a_2, \dots) = L(a_1, a_2, \dots) = \lambda \cdot (a_1, a_2, \dots)$$

let n be the smallest index n with $a_n \neq 0$. Then $a_{n+1} = \lambda \cdot a_n$, $a_{n+2} = \lambda \cdot a_{n+1}$, \dots , so

$$(a_1, a_2, \dots) = (0, \dots, 0, a_n, \lambda a_n, \lambda^2 a_n, \dots)$$

But this is not in ℓ^2 for $|\lambda| = 1$ and $a_n \neq 0$, so λ on the unit circle is *not* an eigenvalue.

For $|\lambda| = 1$, we can make *approximate* λ -eigenvectors for L by

$$v^{[N]} = (1, \lambda, \lambda^2, \dots, \lambda^N, 0, 0, \dots)$$

since

$$(L - \lambda)v^{[N]} = (\lambda, \lambda^2, \dots, \lambda^N, 0, 0, 0, \dots) - \lambda \cdot (1, \lambda, \lambda^2, \dots, \lambda^N, 0, 0, \dots) = (0, 0, \dots, 0, 0, \lambda^{N+1}, 0, 0, \dots)$$

Since

$$\frac{|(L - \lambda)v^{[N]}|}{|v^{[N]}|} = \frac{|\lambda|^{N+1}}{(1 + |\lambda|^2 + \dots + |\lambda|^{2N})^{1/2}} = \frac{1}{\sqrt{N+1}} \longrightarrow 0$$

there can be no continuous $(L - \lambda)^{-1}$. Thus, λ on the unit circle is in the spectrum, but not in the point spectrum.

That the unit circle is in the spectrum also follows from the observation above that all λ with $|\lambda| < 1$ are eigenvalues, and the fact that the spectrum is *closed*.

The spectrum of L is bounded by the operator norm $|L|_{\text{op}}$, and $|L|_{\text{op}}$ is visibly 1, so is nothing *else* in the spectrum.

To see that the unit circle is the *continuous* spectrum of L , as opposed to *residual*, we show that $L - \lambda$ has dense image for $|\lambda| = 1$. Indeed, for w such that, for all $v \in \ell^2$,

$$0 = \langle (L - \lambda)v, w \rangle = \langle v, (L^* - \bar{\lambda})w \rangle = \langle v, (R - \bar{\lambda})w \rangle$$

we would have $(R - \bar{\lambda})w = 0$. However, we have seen that R has no eigenvalues. Thus, $L - \lambda$ always has dense image, and the unit circle is continuous spectrum for L .

Reversing that discussion, every λ with $|\lambda| < 1$ is in the residual spectrum of R , because such λ is not an eigenvalue, and $R - \lambda$ does *not* have dense image: for w a $\bar{\lambda}$ -eigenvector for L ,

$$\langle (R - \lambda)v, w \rangle = \langle v, (R^* - \bar{\lambda})w \rangle = \langle v, (L - \bar{\lambda})w \rangle = \langle v, 0 \rangle = 0$$

That is, the image $(R - \lambda)\ell^2$ is in the orthogonal complement to the eigenvector w . The same computation shows that the unit circle is *continuous* spectrum for R , because it is *not* eigenvalues for L .

[7.2] Volterra operator We will show that the Volterra operator $Vf(x) = \int_0^x f(t) dt$ on $L^2[0, 1]$ is not self-adjoint, that its spectrum is $\{0\}$, and that it has no eigenvalues.

As in the next chapter, since the Volterra operator is given by an L^2 integral kernel, it is *Hilbert-Schmidt*, hence *compact*.

A relation $Tf = \lambda \cdot f$ for $f \in L^2$ and $\lambda \neq 0$ implies f is *continuous*:

$$|\lambda| \cdot |f(x+h) - f(x)| = |Tf(x+h) - Tf(x)| \leq \int_x^{x+h} 1 \cdot |f(t)| dt \leq |h|^{\frac{1}{2}} \cdot |f|_{L^2}$$

The fundamental theorem of calculus would imply f is continuously differentiable and $\lambda \cdot f' = (Tf)' = f$. Thus, f would be a constant multiple of $e^{x/\lambda}$, by the mean value theorem. However, by Cauchy-Schwarz-Bunyakovsky, for a λ -eigenfunction

$$|\lambda| \cdot |f(x)| \leq |x|^{\frac{1}{2}} \cdot |f|_{L^2}$$

No non-zero multiple of the exponential satisfies this. Thus, there are no eigenvectors for *non-zero* eigenvalues.

For $f \in L^2[0, 1]$ and $Tf = 0 \in L^2[0, 1]$, Tf is almost everywhere 0. Since $x \rightarrow Tg(x)$ is unavoidably *continuous*, $Tf(x)$ is 0 for all x . Thus, for all x, y in the interval,

$$0 = 0 - 0 = Tf(y) - Tf(x) = \int_x^y f(t) dt$$

That is, $x \rightarrow Tf(x)$ is orthogonal in $L^2[0, 1]$ to all characteristic functions of intervals. Finite linear combinations of these are dense in $C^o[0, 1]$ in the L^2 topology, and $C^o[0, 1]$ is dense in $L^2[0, 1]$. Thus $f = 0$, and there are no eigenvectors for the Volterra operator.

To see that the spectrum is *at most* $\{0\}$, show that the *spectral radius* is 0:

$$\begin{aligned} T^n f(x) &= \int_0^x \int_0^{x_{n-1}} \dots \int_0^{x_2} \int_0^{x_1} f(t) dt dx_1 \dots dx_{n-1} = \int_0^x f(t) \left(\int_t^x \int_t^{x_{n-1}} \dots \int_t^{x_2} dx_1 \dots dx_{n-1} \right) dt \\ &= \int_0^x f(t) \cdot \frac{(x-t)^{n-1}}{(n-1)!} dt \end{aligned}$$

From this, $|T^n|_{\text{op}} \leq \frac{1}{n!}$, and

$$\begin{aligned} \log \lim_{2n} \left(\frac{1}{(2n)!} \right)^{1/2n} &= -\lim_{2n} \frac{1}{2n} \cdot \log(2n)! = -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq 2n} \log k \\ &= -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq n} (\log k + \log(2n - k + 1)) \leq -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq \frac{n}{k}} (\log k + \log(2n - k + 1)) \\ &\leq -\lim_{2n} \frac{1}{2n} \sum_{1 \leq k \leq \frac{n}{k}} \log 2n = -\lim_{2n} \frac{\log 2n}{2} = -\infty \end{aligned}$$

since $k(2n - k) \geq 2n$ for $1 \leq k \leq n$, noting the sign. That is, $\lim_n |T^n|_{\text{op}}^{1/n} = 0$, so the spectral radius is 0. Since the spectrum is non-empty, it must be exactly $\{0\}$.

8. Weyl's criterion for continuous spectrum

H. Weyl gave a criterion for *continuous* spectrum analogous to the definition of *discrete* spectrum. This criterion is decisive for *normal* operators.

For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) in the Hilbert space V such that $(T - \lambda)v_n \rightarrow 0$ as $n \rightarrow +\infty$ is an *approximate eigenvector* for λ .

[8.1] Theorem: For λ *not* an eigenvalue for T , and for $(T - \lambda)V$ *not closed*, λ is in the spectrum of T if and only if λ has an approximate eigenvector.

[8.2] Remark: This criterion is *not* uniformly reliable for detecting *residual* spectrum, which is why we must impose a further condition. ^[2] For example, we have seen that, for $T : V \rightarrow V$ a *normal* linear operator, for λ in the spectrum but not an eigenvalue, $(T - \lambda)V$ is *dense* in V but is not all of V . Thus, the hypothesis of the theorem is met for normal T . We give an example of failure to detect residual spectrum after the proof.

Proof: Certainly if λ is an eigenvalue, with non-zero eigenvalue v , the constant sequence v, v, v, \dots fits the requirement.

For general spectrum, let $S = T - \lambda$. For v_1, v_2, \dots with $|v_n| = 1$ and $Sv_n \rightarrow 0$, any alleged (continuous ^[3]) S^{-1} would give, interchanging S^{-1} and the limit by continuity,

$$0 = S^{-1}(\lim_n Sv_n) = \lim_n S^{-1}Sv_n = \lim_n v_n$$

contradiction. Thus, existence of an approximate eigenvector for $T - \lambda$ implies that $T - \lambda$ is not invertible.

Conversely, for $S = T - \lambda$ not invertible, but λ not an eigenvalue, then S is *injective* but not *surjective*. We further assume that the image of S is *not closed*. ^[4] In that case, S is injective, not surjective, and

^[2] Recall that *residual* spectrum of T is λ such that $T - \lambda$ is *injective*, but does *not* have dense image.

^[3] Recall that when there is an everywhere-defined, linear inverse S^{-1} to S , necessarily S is a continuous bijection, and by the *open mapping theorem* S is *open*. That is, there is $\delta > 0$ such that $|Sv| \geq \delta \cdot |v|$ for all v . This exactly asserts the boundedness of S^{-1} , so S^{-1} is *continuous*.

^[4] The image is not closed, for example, when T (hence S) has no residual spectrum, which is the case when T (hence S) is *normal*, or *self-adjoint*.

by non-closedness of the image there is v_o (with $|v_o| = 1$) not in the image of S , and v_1, v_2, \dots such that $Sv_1, Sv_2, \dots \rightarrow v_o$. If $\{v_n\}$ were a Cauchy sequence, then it would have a limit, and by continuity of S

$$v_o = \lim_n Sv_n = S(\lim_n v_n)$$

and v_o would be in the image of S , contradicting our assumption. Thus, $\{v_n\}$ is *not* Cauchy. In particular, we can replace $\{v_n\}$ by a subsequence such that there is $\delta > 0$ such that $|v_m - v_n| \geq \delta$ for all $m \neq n$. Then $w_n = v_n - v_{n+1}$ forms an approximate 0-eigenvector, since their lengths are bounded away from 0, and

$$Sw_n = S(v_n - v_{n+1}) = Sv_n - Sv_{n+1} \rightarrow v_o - v_o = 0$$

as desired.

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[8.3] Remark: As noted, the case that λ is not an eigenvector, $T - \lambda$ is not surjective, and/or but the image of $S = T - \lambda$ is *closed*, can only occur for non-normal T . For example, $T : \ell^2 \rightarrow \ell^2$ by

$$T(c_1, c_2, \dots) = (c_1, 0, c_2, 0, c_3, 0, \dots)$$

is injective, not surjective, and has closed image. It is not invertible, but there is no approximate eigenvector for 0, so the criterion fails in this (non-normal) example.
