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01b. Norms and metrics on vectorspaces

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Many natural real or complex vector spaces of functions, such as $C^o[a,b]$ and $C^k[a,b]$, have (several!) natural metrics $d(,)$ ncoming from *norms* $|\cdot|$ by the recipe

$$d(v,w) \;=\; |v-w|$$

A real or complex vector space with an inner product $\langle , \rangle$ always has an associated norm

$$|v| \;=\; \langle v,v\rangle^{\frac{1}{2}}$$

If so, the vector space has much additional geometric structure, including notions of orthogonality and projections. However, very often the natural norm on a vector space of functions does *not* come from an inner product, which creates complications. When the vector space is *complete* with respect to the associated metric, the vector space is a *Banach space.* Abstractly, Banach spaces are less convenient than Hilbert spaces (complete *inner-product* spaces), and normed spaces are less convenient than inner-product spaces.

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1. Normed vector spaces

A real or complex [1] vectorspace V with a non-negative, real-valued function, the *norm*,

$$|\cdot| : V \longrightarrow \mathbb{R}$$

with properties

$$\begin{array}{ll} |x+y| \le |x| + |y| & \text{(triangle inequality)} \\ |\alpha x| = |\alpha|\cdot|x| & (\alpha \text{ real/complex, } x\in V) \\ |x| = 0 \;\;\Rightarrow\;\; x=0 & \text{(positivity)} \end{array}$$

is a *normed (real or complex) vectorspace*, or simply *normed space.* Sometimes a normed space is called *pre-Banach.* Because of the triangle inequality, the associated function

$$d(x,y) \;\;=\;\; |x-y|$$

is a *metric.* The *symmetry* of $d(,)$ comes from a special case of the homogeneity:

$$d(y,x) \;=\; |y-x| \;=\; |(-1)\cdot(x-y)| \;=\; |-1|\cdot|x-y| \;=\; |x-y| \;=\; d(x,y)$$

When V is *complete* with respect to the metric associated to the norm, V is a *Banach space.*

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[1] In fact, for many purposes, the scalars need not be $\mathbb{R}$ or $\mathbb{C}$, need not be locally compact, and need not even be commutative. The basic results hold for Banach spaces over non-discrete, complete, normed division rings. This allows scalars like the p-adic field $\mathbb{Q}_p$, or Hamiltonian quaternions $\mathbb{H}$, and so on.

Metrics on vector spaces arising from norms have some special properties. First, they are degree-one *homogeneous*, in the sense that for scalars α and vectors x, y,

$$d(\alpha \cdot x, \alpha \cdot y) \;=\; |\alpha \cdot x - \alpha \cdot y| \;=\; |\alpha| \cdot |x - y| \;=\; |\alpha| \cdot d(x, y)$$

They are *translation invariant*, in the sense that for and vectors x, y, z,

$$d(x + z, y + z) \;=\; |(x + z) - (y + z)| \;=\; |x - y| \;=\; d(x, y)$$

The basic example of a normed vector space is $C^o[a, b]$, with sup-norm

$$|f|_{\text{sup}} \;=\; |f|_{C^o} \;=\; \sup_{x \in [a,b]} |f(x)|$$

This is part of a larger family of vector spaces of functions: continuously k-fold differentiable functions $C^k[a, b]$, with norm

$$|f|_{C^k} \;=\; |f|_{C^o} + |f'|_{C^o} + |f''|_{C^o} + \ldots + |f^{(k)}|_{C^o}$$

There is at least one other natural choice of norm on $C^k[a, b]$, namely,

$$|f|_{\text{alt}} \;=\; \max_{0 \le j \le k} |f^{(j)}|_{C^o}$$

But these two norms are *comparable*, in the sense that there are constants $0 < A, B < +\infty$ such that, for all $f \in C^k[a, b]$,

$$A \cdot |f|_{C^k} \;\le\; |f|_{\text{alt}} \;\le\; B \cdot |f|_{C^k}$$

In particular, $A = 1$ and $B = k + 1$ work. This implies that a similar comparability holds for the associated metrics, so the associated topologies are *identical.*

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2. *Inner-product spaces and Cauchy-Schwarz-Bunyakowsky inequality*

An *inner-product space* or *pre-Hilbert space* is a (real or) complex[2] vector space V with $\langle , \rangle : V \times V \to$ scalars with properties

$$\begin{cases} \langle x, y \rangle & = & \overline{\langle y, x \rangle} & \text{(the } \textit{hermitian-symmetric} \text{ property)} \\ \langle x + x', y \rangle & = & \langle x, y \rangle + \langle x', y \rangle & (\textit{additivity in} \text{ first argument}) \\ \langle x, y + y' \rangle & = & \langle x, y \rangle + \langle x, y' \rangle & (\textit{additivity} \text{ in second argument}) \\ \langle x, x \rangle & \ge & 0 & \text{(and equality only for } x = 0\text{: } \textit{positivity}) \\ \langle \alpha x, y \rangle & = & \alpha \langle x, y \rangle & (\textit{linearity} \text{ in first argument}) \\ \langle x, \alpha y \rangle & = & \bar{\alpha} \langle x, y \rangle & (\textit{conjugate-linearity} \text{ in second argument}) \end{cases}$$

Among other easy consequences of these requirements, for all $x, y \in V$

$$\langle x, 0 \rangle \;=\; \langle 0, y \rangle \;=\; 0$$

where inside the angle-brackets the 0 is the zero-*vector*, and outside it is the zero-*scalar.*

The *associated norm* $|\;|$ on V is

$$|x| \;=\; \langle x, x \rangle^{1/2}$$

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[2] Although *normed* spaces make sense over a wider range of scalars than just real or complex, *inner-product* spaces essentially need $\mathbb{R}$ or $\mathbb{C}$.

of course with the non-negative square-root. Even though we use the same notation for the norm on V as for the usual complex value $||$, context will make clear which is meant. The triangle inequality follows from the *Cauchy-Schwarz-Bunyakowsky inequality* just below.

Most norms on Banach spaces do *not* arise from inner products. Norms arising from inner products recover the inner product via the *polarization* identities

$$4\langle x, y\rangle \;=\; |x+y|^2 \;-\; |x-y|^2 \qquad \text{(real vector space)}$$

$$4\langle x, y\rangle \;=\; |x+y|^2 \;-\; |x-y|^2 \;+\; i|x+iy|^2 \;-\; i|x-iy|^2 \qquad \text{(complex vector space)}$$

Given a norm on a vector space, *if* the polarization expression gives an inner product, *then* the norm is produced by that inner product. However, checking whether the polarization expression is bilinar or hermitian, may be awkward or non-intuitive.

For two vectors v, w in a pre-Hilbert space, if $\langle v, w\rangle = 0$ then v, w are *orthogonal* or *perpendicular*, sometimes written $v \perp w$. A vector v is a *unit vector* if $|v| = 1$.

[2.1] Theorem: *(Cauchy-Schwarz-Bunyakowsky inequality)*

$$|\langle x, y\rangle| \;\leq\; |x| \cdot |y|$$

with strict inequality unless x, y are collinear, i.e., unless one of x, y is a multiple of the other.

Proof: Suppose that x is not a scalar multiple of y, and that neither x nor y is 0. Then $x - \alpha y$ is not 0 for any complex α. Consider

$$0 \;<\; |x - \alpha y|^2$$

We know that the inequality is indeed *strict* for all α since x is not a multiple of y. Expanding this,

$$0 \;<\; |x|^2 - \alpha\langle x, y\rangle - \bar{\alpha}\langle y, x\rangle + \alpha\bar{\alpha}|y|^2$$

Let

$$\alpha \;=\; \mu t$$

with real t and with $|\mu| = 1$ so that

$$\mu\langle x, y\rangle \;=\; |\langle x, y\rangle|$$

Then

$$0 \;<\; |x|^2 - 2t|\langle x, y\rangle| + t^2|y|^2$$

The *minimum* of the right-hand side, viewed as a function of the real variable t, occurs when the derivative vanishes, i.e., when

$$0 \;=\; -2|\langle x, y\rangle| + 2t|y|^2$$

Using this minimization as a *mnemonic* for the value of t to substitute, we indeed substitute

$$t \;=\; \frac{|\langle x, y\rangle|}{|y|^2}$$

into the inequality to obtain

$$0 \;<\; |x|^2 + \left(\frac{|\langle x, y\rangle|}{|y|^2}\right)^2 \cdot |y|^2 - 2\frac{|\langle x, y\rangle|}{|y|^2} \cdot |\langle x, y\rangle|$$

which simplifies to

$$|\langle x, y\rangle|^2 \;<\; |x|^2 \cdot |y|^2$$

as desired. ///

[2.2] Corollary: *(Triangle inequality)* For v, w in an inner-product space V, we have $|v + w| \leq |v| + |w|$. Thus, with distance function $d(v, w) = |v - w|$, we have the triangle inequality

$$d(x,z) \;=\; |x-z| \;=\; |(x-y)+(y-z)| \;\leq\; |x-y|+|y-z| \;=\; d(x,y)+d(y,z)$$

Proof: Squaring and expanding, noting that $\langle v, w\rangle + \langle w, v\rangle = 2\mathrm{Re}\,\langle v, w\rangle$,

$$(|v|+|w|)^2-|v+w|^2 \;=\; \Big(|v|^2+2|v|\cdot|w|+|w|^2\Big)-\Big(|v|^2+\langle v,w\rangle+\langle w,v\rangle+|w|^2\Big) \;\geq\; 2|v|\cdot|w|-2|\langle v,w\rangle| \;\geq\; 0$$

giving the asserted inequality. ///

All inner product spaces are normed spaces, but many natural normed spaces are *not* inner-product, and may fail to have useful properties. The same holds for Hilbert and Banach spaces.

An iconic example of an inner-product space is $C^o[a, b]$ with inner product

$$\langle f,g\rangle \;=\; \int_a^b f(x)\cdot\overline{g(x)}\,dx \;=\; \int f\cdot\overline{g}$$

With the associated metric $C^o[a, b]$ is *not* complete. The completion of $C^o[a, b]$ with respect to that metric is called $L^2[a, b]$.

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3. *Normed spaces of linear maps*

There is a *natural norm* on the vector space of continuous linear maps $T : X \to Y$ from one normed vector space X to another normed vector space Y. Even when X, Y are Hilbert spaces, the set of continuous linear maps $X \to Y$ is generally only a *Banach* space.

Let $\mathrm{Hom}^o(X, Y)$ denote [3] the collection of continuous linear maps from the normed vectorspace X to the normed vectorspace Y. We may use the same notation $|\cdot|$ for the norms on both X and Y, since context will make clear which is which.

A linear (not necessarily continuous) map $T : X \to Y$ from one normed space to another has *uniform operator norm*

$$|T| \;=\; |T|_{\text{uniform}} \;=\; \sup_{|x|_X\leq 1}|Tx|_Y \;=\; \sup_{|x|\leq 1}|Tx|$$

where we momentarily allow the value $+\infty$. Such T is called *bounded* if $|T| < +\infty$. There are several obvious variants of the expression for the uniform norm:

$$|T| \;=\; \sup_{|x|\leq 1}|Tx| \;=\; \sup_{|x|<1}|Tx| = \sup_{|x|\neq 0}\frac{|Tx|}{|x|}$$

[3.1] Theorem: For a linear map $T : X \to Y$ from one normed space to another, the following conditions are equivalent:
- T is *continuous.*
- T is *continuous at* 0.
- T is *bounded.*

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[3] Another traditional notation for the collection of continuous linear maps from X to Y is $B(X, Y)$, where B stands for *bounded*. But the Hom notation fits better into a larger pattern of notational conventions.

Proof: First, show that continuity *at a point* x_o implies continuity *everywhere*. For another point x_1, given $\varepsilon > 0$, take $\delta > 0$ so that $|x - x_o| < \delta$ implies $|Tx - Tx_o| < \varepsilon$. Then for $|x' - x_1| < \delta$

$$|(x' + x_o - x_1) - x_o| \;<\; \delta$$

By linearity of T,

$$|Tx' - Tx_1| \;=\; |T(x' + x_o - x_1) - Tx_o| \;<\; \varepsilon$$

which is the desired continuity at x_1.

Now suppose that T is continuous at 0. For $\varepsilon > 0$ there is $\delta > 0$ so that $|x| < \delta$ implies $|Tx| < \varepsilon$. For $x \neq 0$,

$$\Big|\frac{\delta}{2|x|}x\Big| \;<\; \delta$$

so

$$\Big|T\frac{\delta}{2|x|}\cdot x\Big| \;<\; \varepsilon$$

Multiplying out and using the linearity, boundedness is obtained:

$$|Tx| \;<\; \frac{2\varepsilon}{\delta}\cdot|x|$$

Finally, prove that boundedness implies continuity at 0. Suppose there is C such that $|Tx| < C|x|$ for all x. Then, given $\varepsilon > 0$, for $|x| < \varepsilon/C$

$$|Tx| < C|x| < C\cdot\frac{\varepsilon}{C} \;=\; \varepsilon$$

which is continuity at 0. ///

The space $\mathrm{Hom}^o(X,Y)$ of continuous linear maps from one normed space X to another normed space Y has a natural structure of vectorspace by

$$(\alpha T)(x) \;=\; \alpha\cdot(Tx) \qquad \text{and} \qquad (S+T)x \;=\; Sx + Tx$$

for $\alpha \in \mathbb{C}$, $S, T \in \mathrm{Hom}^o(X,Y)$, and $x \in X$.

[3.2] Proposition: With the uniform operator norm, the space $\mathrm{Hom}^o(X,Y)$ of continuous linear operators from a normed space X to a *Banach* space Y is *complete*, whether or not X itself is complete. Thus, $\mathrm{Hom}^o(X,Y)$ is a Banach space.

Proof: Let $\{T_i\}$ be a Cauchy sequence of continuous linear maps $T : X \to Y$. Try defining the limit operator T in the natural fashion, by

$$Tx \;=\; \lim_i Tx_i$$

First, check that this limit exists. Given $\varepsilon > 0$, take i_o large enough so that $|T_i - T_j| < \varepsilon$ for $i, j > i_o$. By the definition of the uniform operator norm,

$$|T_i x - T_j x| \;<\; |x|\varepsilon$$

Thus, the sequence of values $T_i x$ is Cauchy in Y, so has a limit in Y. Call the limit Tx.

We need to prove that the map $x \to Tx$ is *continuous* and *linear*. The arguments are inevitable. Given $c \in \mathbb{C}$ and $x \in X$, for given $\varepsilon > 0$ choose index i so that for $j > i$ both $|Tx - T_j x| < \varepsilon$ and $|Tcx - T_j cx| < \varepsilon$. Then

$$|Tcx - cTx| \leq |Tcx - T_j cx| + |cT_j x - cTx| = |Tcx - T_j cx| + |c|\cdot|T_j x - Tx| < (1+|c|)\varepsilon$$

This is true for every ε, so $Tcx = cTx$. Similarly, given $x, x' \in X$, for $\varepsilon > 0$ choose an index i so that for $j > i$ $|Tx - T_j x| < \varepsilon$ and $|Ty - T_j y| < \varepsilon$ and $|T(x+y) - T_j(x+y)| < \varepsilon$. Then

$$|T(x+y) - Tx - Ty| \le |T(x+y) - T_j(x+y)| + |T_j x - Tx| + |T_j y - Ty| < 3\varepsilon$$

This holds for every ε, so $T(x+y) = Tx + Ty$.

For continuity, show that T is *bounded*. Choose an index i_o so that for $i, j \ge i_o$

$$|T_i - T_j| \le 1$$

This is possible since the sequence of operators is Cauchy. For such i, j

$$|T_i - T_j x| \;\le\; |x|$$

for all x. Thus, for $i \ge i_o$

$$|T_i x| \;\le\; |(T_i - T_{i_o})x| + |T_{i_o} x| \;\le\; |x|(1 + |T_{i_o}|)$$

Taking a limsup,

$$\limsup_i |T_i x| \;\le\; |x|(1 + |T_{i_o}|)$$

This implies that T is bounded, and so is continuous.

Finally, we should see that $Tx = \lim_i T_i x$ is the operator-norm limit of the T_i. Given $\varepsilon > 0$, let i_o be sufficiently large so that $|T_i x - T_j x| < \varepsilon$ for all $i, j \ge i_o$ and for all $|x| \le 1$. Then $|Tx - Tx_i| \le \varepsilon$ and

$$\sup_{|x|\le 1} |Tx - T_i x| \;\le\; \sup_{|x|\le 1} \varepsilon \;=\; \varepsilon$$

giving the desired outcome. ///

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4. *Dual spaces of normed spaces*

This section considers an important special case of continuous linear maps between normed spaces, namely continuous linear maps from Banach spaces to *scalars*. All assertions are special cases of those for continuous linear maps to general Banach spaces, but deserve special attention.

For X a normed vectorspace with norm $|\,|$, a *continuous* linear map $\lambda : X \to \mathbb{C}$ is a *(continuous linear) functional* on X. Let

$$X^* \;=\; \mathrm{Hom}^o(X, \mathbb{C})$$

denote the collection of all such (continuous) functionals.

As more generally, for *any* linear map $\lambda : X \to \mathbb{C}$ of a normed vectorspace to $\mathbb{C}$, the *norm* $|\lambda|$ is

$$|\lambda| \;=\; \sup_{|x|\le 1} |\lambda x|$$

where $|\lambda x|$ is the absolute value of the value $\lambda x \in \mathbb{C}$. We allow the value $+\infty$. Such a linear map λ is *bounded* if $|\lambda| < +\infty$.

As a special case of the corresponding general result:

[4.1] Corollary: For a k-linear map $\lambda : X \to k$ from a normed space X to k, the following conditions are equivalent:

- The map λ is *continuous*.

- The map λ is *continuous at one point.*
- The map λ is *bounded.*

Proof: These are special cases of the earlier proposition where the target was a general Banach space. ///

The *dual space*

$$X^* \;=\; \mathrm{Hom}^o(X, \mathbb{C})$$

of X is the collection of *continuous* linear functionals on X. This dual space has a natural structure of vectorspace by

$$(\alpha\lambda)(x) \;=\; \alpha \cdot (\lambda x) \qquad\text{and}\qquad (\lambda + \mu)x \;=\; \lambda x + \mu x$$

for $\alpha \in \mathbb{C}$, $\lambda, \mu \in X^*$, and $x \in X$. It is easy to check that the norm

$$|\lambda| \;=\; \sup_{|x|\le 1} |\lambda x|$$

really is a norm on X^*, in that it meets the conditions

- *Positivity:* $|\lambda| \ge 0$ with equality only if $\lambda = 0$.
- *Homogeneity:* $|\alpha\lambda| = |\alpha| \cdot |\lambda|$ for $\alpha \in k$ and $\lambda \in X^*$. As a special case of the discussion of the uniform norm on linear maps, we have

[4.2] Corollary: The dual space X^* of a normed space X, with the natural norm, is a Banach space. That is, with respect to the natural norm on continuous functionals, it is *complete.* ///

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5. *Extensions by continuity*

The following is a very general and important pattern. In particular, it is often applied in situations where the underlying *set* of X is the same as the underlying set of Y, but the norms are different.

[5.1] Theorem: Let $T : X \to Y$ be a continuous linear map from one normed space to another. Let $\widetilde{X}$ be the completion of X, and $\widetilde{Y}$ the completion of Y, both with respect to the metrics attached to their respective norms. Then a map $\widetilde{T} : \widetilde{X} \to \widetilde{Y}$ apparently given by

$$\widetilde{T}(\widetilde{X} - \lim_n x_n) \;=\; \widetilde{Y} - lim_n T(x_n)$$

is a well-defined, continuous, linear map, and $\widetilde{T}|_X = T$.

Proof: To prove that $\widetilde{T}$ is well-defined, by subtraction is suffices to show that if $\widetilde{X} - \lim x_n = 0$, then $\widetilde{Y} - \lim T(x_n) = 0$. Indeed, since T is continuous, it is *bounded* in the sense that there is $0 < C < +\infty$ such that $|Tx_n| \le C \cdot |x|$ for all $x \in X$. Since $x_n \to 0$, for all $\varepsilon > 0$ there is n_o such that $n \ge n_o$ implies $|x_n| < \varepsilon$. For such n, $|Tx_n| \le C \cdot \varepsilon$. Thus, $Tx_n \to 0$ in Y, hence in $\widetilde{Y}$, and $\widetilde{T}$ is well-defined.

In fact, the previous also shows that $\widetilde{T}$ is bounded with any bound that applies to T, so $\widetilde{T}$ is continuous.

Since limits are homogeneous (degree one) and linear, the obvious computations will show that $\widetilde{T}$ is linear. Just to be clear: for scalar α and vectors $x_n \in X$,

$$\widetilde{T}(\alpha \cdot \lim x_n) \;=\; \widetilde{T}(\lim \alpha \cdot x_n) \;=\; \lim T(\alpha \cdot x_n) \;=\; \lim \alpha \cdot T(x_n) \;=\; \alpha \cdot \lim T(x_n) \;=\; \alpha \cdot \widetilde{T}(\lim_n x)$$

and for $x_n, x_n' \in X$,

$$\widetilde{T}(\lim x_n + \lim x'_n) \;=\; \widetilde{T}(\lim(x_n + x'_n)) \;=\; \lim T(x_n + x'_n)$$
$$=\; \lim(T(x_n) + T(x'_n)) \;=\; \lim T(x_n) + \lim T(x'_n) \;=\; \widetilde{T}(\lim x_n) + \widetilde{T}(\lim x'_n)$$

For $x \in X \subset \widetilde{X}$, we can take the constant sequence $x_n = x$, and

$$\widetilde{T}(x) \;=\; \widetilde{T}(\lim x_n) \;=\; \lim T(x_n) \;=\; \lim T(x) \;=\; T(x)$$

so $\widetilde{T}$ does simply extend T to the completions. ///