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04b. Product measures and Fubini-Tonelli theorem

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1. *Product measures, completions of measures*

Let X, μ and Y, ν be measure spaces with corresponding σ-algebras A, B. Assume X and Y are σ-*finite*, in the sense that they are countable unions of finite-measure sets.

First, the *product* σ-algebra is the σ-algebra in $X \times Y$ generated by all products $E \times F$ with $E \in A$ and $F \in B$.

For iterated integrals to make sense, we need to check a few things. For $E \in A \times B$, for $x \in X$ and $y \in Y$, let

$$E_x \;=\; \{y \in Y : (x,y) \in E\} \qquad \text{and} \qquad E^y \;=\; \{x \in X : (x,y) \in E\}$$

As a consistency check, we have

[1.1] Theorem: For $E \in A \times B$, for $x \in X$ and $y \in Y$, $E_x \in A$ and $E^y \in B$. The function $x \to \nu(E_x)$ is μ-measurable, $y \to \mu(E^y)$ is ν-measurable, and

$$\int_X \nu(E_x)\, d\mu(x) \;=\; \int_Y \mu(E^y)\, d\nu(y)$$

Proof: [... iou ...] ///

Then the *product measure* $\mu \times \nu$ can be defined in the expected fashion: for $E \in A \times B$,

$$(\mu \times \nu)(E) \;=\; \int_X \nu(E_x)\, d\mu(x) \;=\; \int_Y \mu(E^y)\, d\nu(y)$$

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2. *Fubini-Tonelli theorem(s)*

Let X, μ and Y, ν be measure spaces with corresponding σ-algebras A, B. Assume X and Y are σ-*finite*.

[2.1] Theorem: *(Fubini-Tonelli)* For complex-valued measurable f, g, if any one of

$$\int_X \int_Y |f(x,y)|\, d\mu(x)\, d\nu(y) \qquad \int_Y \int_X |f(x,y)|\, d\nu(y)\, d\mu(x) \qquad \int_{X\times Y} |f(x,y)|\, d\mu{\times}\nu$$

is finite, then they *all* are finite, and are equal. For $[0, +\infty]$-valued functions f,

$$\int_X \int_Y f(x,y)\, d\mu(x)\, d\nu(y) \;=\; \int_Y \int_X f(x,y)\, d\nu(y)\, d\mu(x) \;=\; \int_{X\times Y} f(x,y)\, d\mu{\times}\nu$$

although the values may be $+\infty$.

Proof: [... iou ...] ///

To explain what the *product measure* $\mu \times \nu$ should be, and also for a proof of the theorem, we need the notion of *monotone class*. A monotone class in a set X is a set $\mathcal{M}$ of subsets of X closed under countable ascending unions and under countable descending intersections. That is, if

$$M_1 \subset M_2 \subset M_3 \subset \ldots$$

$$N_1 \supset N_2 \supset N_3 \supset \ldots$$

are collections of sets in $\mathcal{M}$, then

$$\bigcup_i M_i \qquad \bigcap_i N_i$$

both lie in $\mathcal{M}$, as well. Another characterization of $\mathcal{A} \times \mathcal{B}$ is that it is the smallest monotone class containing all products $E \times F$ with $E \in \mathcal{A}$ and $F \in \mathcal{B}$.

Let f be a $\mathcal{A} \times \mathcal{B}$-measurable function on $X \times Y$. (Note that this does not depend upon having a 'product measure', but only upon the sigma-algebra!) Then all the functions

$$x \to f(x, y) \qquad \text{(for fixed } y \in Y$$

$$y \to f(x, y) \qquad \text{(for fixed } x \in X$$

are measurable (in appropriate senses). In particular, we could apply this to the *characteristic function* of a set $G \in \mathcal{A} \times \mathcal{B}$.

Now we come to the point where the sigma-finiteness of X and Y is necessary. For $G \in \mathcal{A} \times \mathcal{B}$, let

$$f(x) = \nu(G_x) \qquad g(y) = \mu(G_y)$$

where G_x, G_y are as above. We have already noted that f, g are *measurable*. Further,

$$\int_X f(x)\, d\mu(x) = \int_Y g(y)\, d\nu(y)$$

This is proven by showing that the collection of G for which the conclusion is true is a *monotone class* containing all products $E \times F$.

In light of the latter equality, we can define the *product measure* $\mu \times \nu$ on $G \in \mathcal{A} \times \mathcal{B}$ by

$$(\mu \times \nu)(G) = \int_X f(x)\, d\mu(x) = \int_Y g(y)\, d\nu(y)$$

with notation as just above. The *countable additivity* follows from a preliminary version of Fubini's theorem, namely that if f_i are countably-many $[0, +\infty]$-valued functions on a measure space Ω, then

$$\int_\Omega \sum_i f_i = \sum_i \int_\Omega f_i$$

which itself is a little corollary of the monotone convergence theorem.

sectionCompletions of measures

For example, a reasonable measure on $\mathbb{R}^m \times \mathbb{R}^n$ should include many sets not expressible as countable unions of products $E \times F$ where $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^n$. For example, diagonal subsets of the form $D = \{(x, x) : 0 \le x \le 1\} \subset \mathbb{R}^2$ are not countable unions of products, but should surely be measurable.

One way to accomplish this is by *completion* of the product measure.

Then the *completion* of $\mu \times \nu$ further assigns measure 0 to *any* subset S of $T \in A \times B$ with $(\mu \times \nu)(T) = 0$, and adjoins all such sets to the σ-algebra $A \times B$.

[2.2] Claim: Lebesgue measure on $\mathbb{R}^m \times \mathbb{R}^n$ is the completion of the product of Lebesgue measures on $\mathbb{R}^m$ and $\mathbb{R}^n$.

Proof: [... iou ...] ///

Completing a product measure is usually what we want, but it slightly complicates the statement of the corresponding Fubini-Tonelli theorem:

[2.3] Theorem: Let X, A, μ and Y, B, ν be *complete* measure spaces, with X, Y σ-finite. Let f be a function on $X \times Y$ measurable with respect to the *completion* of the product measure. Then $x \to f(x, y)$ and $y \to f(x, y)$ are μ-measurable and ν-measurable (only) *almost everywhere.*

Proof: [... iou ...] ///

[2.4] Remark: To be precise, *completeness* is a property of σ-algebras, not of measures.
