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04c. Riesz-Markov-Kakutani theorem, Lebesgue measure

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1. Riesz-Markov-Kakutani theorem and regularity

Let X be a locally compact, Hausdorff, topological space. A map $f \to \lambda(f)$ of continuous, compactly supported functions $C_c^o(X)$ to scalars is *positive* when $\lambda(f) \geq 0$ for $f \in C_c^o(X)$ taking values in $[0, +\infty)$.

[1.1] Theorem: *(Riesz, Markov, Kakutani, independently)* Given a positive functional λ on $C_c^o(X)$, there is a σ-algebra A containing all Borel sets, and a positive measure μ on A, such that

$$\lambda(f) \;=\; \int_X f(x)\, d\mu(x) \qquad\qquad (\text{for all } f \in C_c^o(X))$$

• *Outer regularity* holds unconditionally, namely, that for $E \in A$, $\mu(E) \inf_{U \supset E} \mu(U)$ where U ranges over *open* sets containing E.
• *Inner regularity* is conditional: for open E, and for $\mu(E) < \infty$, $\mu(E) = \sup_{K \subset E} \mu(K)$ where K ranges over *compact* sets contained in E.
• μ is *complete*, in the sense that $E' \subset E \in A$ and $\mu(E) = 0$ implies that $E' \in A$.

Proof: (Standard... *[... iou ...]*) ///

With a further mild assumption on the physical space X, including familiar spaces such as $\mathbb{R}^n$, in fact we have unconditional *regularity*:

[1.2] Theorem: Suppose further that X is σ*-compact*, meaning that it is a countable union of compact subsets. Then, in the situation of the previous theorem, μ is unconditionally *inner regular*: $\mu(E) = \sup_{K \subset E} \mu(K)$ as K ranges over compacts contained in E. Thus, the measure μ is a *positive, regular, Borel measure*.

Proof: (Standard... *[... iou ...]*) ///

2. Lebesgue measure

As a corollary of the Riesz-Markov-Kakutani theorem we have a different description of the Lebesgue measure and integral, as an extension of the Riemann integral, with the very useful side effect of proving inner and outer regularity.

In the Riesz-Markov-Kakutani theorem, take $X = \mathbb{R}^n$, and $\lambda(f)$ to be the usual Riemann integral for $f \in C_c^o(\mathbb{R}^n)$, and let Lebesgue measure be the associated *positive, regular, Borel* measure. With this description of Lebesgue measure, as opposed to the more tangible (but also more awkward) Lebesgue outer measure, we must verify that all the expected properties do hold.

[2.1] Corollary: Let μ be Lebesgue measure, induced by the Riesz-Markov-Kakutani theorem from the Riemann integral on $C_c^o(\mathbb{R}^n)$.

• μ is *translation-invariant* in the sense that $\mu(E+x) = \mu(E)$ for all $x \in \mathbb{R}^n$.

• The Lebesgue measure of a cube $(a_1, b_1) \times \ldots \times (a_n, b_n)$ is the product $\prod_i |b_i - a_i|$, and similarly for closed and half-open intervals and their products.

Proof: (Standard... *[... iou ...]*) ///
