

05. L^p spaces, convexity, basic inequalities

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1. Examples: spaces L^p
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1. Examples: spaces L^p

Given a measure space X , for $1 \leq p < \infty$ the usual L^p spaces are

$$L^p(X) = \{\text{measurable } f : |f|_{L^p} < \infty\} \text{ modulo } \sim$$

with the usual L^p norm

$$|f|_{L^p} = \left(\int_X |f|^p \right)^{1/p}$$

and associated metric

$$d(f, g) = |f - g|_{L^p}$$

taking the quotient by the equivalence relation

$$f \sim g \text{ if } f - g = 0 \text{ off a set of measure } 0$$

[1.1] **Remark:** These L^p functions have inevitably ambiguous pointwise values, in conflict with the naive formal definition of *function*.

A simple instance of this construction is

$$\ell^p = \{\text{complex sequences } \{c_i\} \text{ with } \sum_i |c_i|^p < \infty\}$$

with norm $|(c_1, c_2, \dots)|_{\ell^p} = (\sum_i |c_i|^p)^{1/p}$. The analogue of the following theorem for ℓ^p is more elementary.

[1.2] **Theorem:** The space $L^p(X)$ is a complete metric space.

[1.3] **Remark:** In fact, as used in the proof, a Cauchy sequence f_i in $L^p(X)$ has a subsequence converging *pointwise* off a set of measure 0 in X .

Proof: The triangle inequality here is *Minkowski's inequality*. To prove completeness, choose a subsequence f_{n_i} such that

$$|f_{n_i} - f_{n_{i+1}}|_p < 2^{-i}$$

and put

$$g_n(x) = \sum_{1 \leq i \leq n} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and

$$g(x) = \sum_{1 \leq i < \infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

The infinite sum is not necessarily claimed to converge to a finite value for every x . The triangle inequality shows that $|g_n|_p \leq 1$. *Fatou's Lemma* asserts that for $[0, \infty]$ -valued measurable functions h_i

$$\int_X \left(\liminf_i h_i \right) \leq \liminf_i \int_X h_i$$

Thus, $|g|_p \leq 1$, so is finite. Thus,

$$f_{n_1}(x) + \sum_{i \geq 1} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges for almost all $x \in X$. Let $f(x)$ be the sum at points x where the series converges, and on the measure-zero set where the series does not converge put $f(x) = 0$. Certainly

$$f(x) = \lim_i f_{n_i}(x) \quad (\text{for almost all } x)$$

Now prove that this almost-everywhere pointwise limit is the L^p -limit of the original sequence. For $\varepsilon > 0$ take N such that $|f_m - f_n|_p < \varepsilon$ for $m, n \geq N$. Fatou's lemma gives

$$\int |f - f_n|^p \leq \liminf_i \int |f_{n_i} - f_n|^p \leq \varepsilon^p$$

Thus $f - f_n$ is in L^p and hence f is in L^p . And $|f - f_n|_p \rightarrow 0$. ///

[1.4] Theorem: For a locally compact Hausdorff topological space X with positive regular Borel measure μ , the space $C_c^0(X)$ of compactly-supported continuous functions is *dense* in $L^p(X, \mu)$.

Proof: From the definition of *integral* attached to a measure, an L^p function is approximable in L^p metric by a *simple* function, that is, a measurable function assuming only finitely-many values. That is, a simple function is a *finite* linear combination of characteristic functions of measurable sets E . Thus, it suffices to approximate characteristic functions of measurable sets by continuous functions. The assumed *regularity* of the measure gives compact K and open U such that $K \subset E \subset U$ and $\mu(U - E) < \varepsilon$, for given $\varepsilon > 0$. Urysohn's lemma says that there is continuous f identically 1 on K and identically 0 off U . Thus, f approximates the characteristic function of E . ///

[1.5] Corollary: For locally compact Hausdorff X with regular Borel measure μ , $L^p(X, \mu)$ is the L^p -metric completion of $C_c^0(X)$, the compactly-supported continuous functions. ///

[1.6] Remark: *Defining* $L^p(X, \mu)$ to be the L^p completion of $C_c^0(X)$ avoids discussion of ambiguous values on sets of measure zero.

2. Convexity and inequalities

A function f on an interval $(a, b) \subset \mathbb{R}$ is *convex* when its graph bends upward, in the sense that a line segment connecting two points on the graph lies *above* the graph. That is,

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y) \quad (\text{for } 0 \leq t \leq 1 \text{ and } a < x < y < b)$$

The prototype is the exponential function $x \rightarrow e^x$.

[2.1] Claim: Convex \mathbb{R} -valued functions on an open interval (a, b) (allowing $a = -\infty$ and/or $b = +\infty$) are *continuous*.

Proof: Let g be continuous on (a, b) and take $x \in (a, b)$. Fix any s, t such that $a < s < x < t < b$. For y in the range $x < y < t$, the point $(y, g(y))$ is on or above the line through $(s, g(s))$ and $(x, g(x))$, and is below the line through $(x, g(x))$ and $(t, g(t))$, so $g(y) \rightarrow g(x)$ as $y \rightarrow x^+$. For $s < y < x$, the same argument gives *left*-continuity. ///

[2.2] Theorem: (*Jensen's inequality*) Let X be a measure space with positive measure of total measure 1. Let $f \in L^1(X)$ be an \mathbb{R} -valued function on X with $a < f(x) < b$ for all $x \in X$, where a, b can also be $-\infty$ and $+\infty$. For convex g on (a, b) ,

$$g\left(\int_X f\right) \leq \int_X g \cdot f$$

Proof: First, $a < f(x) < b$ gives $a < \int_X f < b$. The convexity condition can be rewritten as the condition that slopes of secants increase from left to right. Thus, for example,

$$\frac{g(y) - g(x)}{y - x} \leq \frac{g(z) - g(y)}{z - y} \quad (\text{for } x < y < z \text{ inside } (a, b))$$

Applying this with $y = \int_X f$,

$$\frac{g(\int_X f) - g(x)}{\int_X f - x} \leq \frac{g(z) - g(\int_X f)}{z - \int_X f} \quad (\text{for all } a < x < \int_X f \text{ and for all } \int_X f < z < b)$$

With

$$S = \sup_{a < x < \int_X f} \frac{g(\int_X f) - g(x)}{\int_X f - x}$$

we have

$$\frac{g(\int_X f) - g(x)}{\int_X f - x} \leq S \leq \frac{g(z) - g(\int_X f)}{z - \int_X f} \quad (\text{for all } a < x < \int_X f \text{ and for all } \int_X f < z < b)$$

Thus, from the left half of the latter inequality,

$$g(x) \geq g(\int_X f) + S \cdot (x - \int_X f) \quad (\text{for } a < x < \int_X f)$$

and from the right half

$$g(z) \geq g(\int_X f) + S \cdot (z - \int_X f) \quad (\text{for } \int_X f < z < b)$$

Thus,

$$g(w) \geq g(\int_X f) + S \cdot (w - \int_X f) \quad (\text{for all } w \text{ in the range } a < w < b)$$

In particular, letting $w = f(x)$ now with $x \in X$,

$$g(f(x)) \geq g(\int_X f) + S \cdot (f(x) - \int_X f) \quad (\text{for all } w \text{ in the range } a < w < b)$$

Since the convex function g is *continuous*, $g \circ f$ is measurable. Integrating in $x \in X$, using the fact that the total measure is 1,

$$\int_X g \circ f \geq g(\int_X f) + S \cdot (\int_X f - \int_X f) = g(\int_X f) + S \cdot 0$$

as claimed. ///

[2.3] **Corollary:** (*Arithmetic-geometric mean inequality*) For positive real numbers a_1, \dots, a_n ,

$$(a_1 a_2 \dots a_n)^{1/n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}$$

Proof: In Jensen's inequality, take $g(x) = e^x$, take X a finite set with n (distinct) elements $\{x_1, \dots, x_n\}$, with each point having measure $1/n$, and $f(x_i) = \log a_i$. Jensen's inequality gives

$$\exp\left(\frac{\log a_1 + \dots + \log a_n}{n}\right) \leq \frac{e^{\log a_1} + \dots + e^{\log a_n}}{n}$$

which gives the assertion. ///

Conjugate exponents are numbers $p, q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

For example, p and $\frac{p}{p-1}$ are conjugate exponents.

Generalizing the Cauchy-Schwarz-Bunyakovsky inequality,

[2.4] Corollary: (Hölder) For conjugate exponents p, q and $[0, +\infty]$ -valued measurable functions f, g ,

$$\int_X f \cdot g \leq \left(\int_X f^p \right)^{\frac{1}{p}} \cdot \left(\int_X g^q \right)^{\frac{1}{q}}$$

Proof: The assertion is trivial if either integral on the right-hand side is $+\infty$ or 0, so suppose the two quantities

$$I = \left(\int_X f^p \right)^{\frac{1}{p}} \quad J = \left(\int_X g^q \right)^{\frac{1}{q}}$$

are finite and non-zero. Renormalize by taking $\varphi = f/I$ and $\psi = g/J$, so that $\int \varphi^p = 1 = \int \psi^q$. For $x \in X$ with $0 < \varphi(x) < \infty$ and $0 < \psi(x) < \infty$, there are real numbers u, v such that $e^{u/p} = \varphi(x)$ and $e^{v/q} = \psi(x)$. Invoking Jensen's inequality on a measure space with just two points with measures $\frac{1}{p}$ and $\frac{1}{q}$, using the convexity of the exponential function,

$$\varphi(x)\psi(x) = e^{\frac{u}{p} + \frac{v}{q}} \leq \frac{e^u}{p} + \frac{e^v}{q} = \frac{\varphi(x)^p}{p} + \frac{\psi(x)^q}{q}$$

Integrating,

$$\int_X \varphi \cdot \psi \leq \int_X \frac{\varphi(x)^p}{p} + \frac{\psi(x)^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

From the renormalization, we are done. ///

For the triangle inequality in L^p spaces for general p , we need

[2.5] Corollary: (Minkowski) For $1 < p < +\infty$ and $[0, +\infty]$ -valued measurable functions f, g ,

$$\left(\int_X (f+g)^p \right)^{\frac{1}{p}} \leq \left(\int_X f^p \right)^{\frac{1}{p}} + \left(\int_X g^p \right)^{\frac{1}{p}}$$

Proof: We prove Minkowski's inequality from Hölder's, using the conjugate exponents p and $q = \frac{p}{p-1}$.

$$\begin{aligned} \int (f+g)^p &= \int f \cdot (f+g)^{p-1} + \int g \cdot (f+g)^{p-1} \\ &\leq \left(\int f^p \right)^{\frac{1}{p}} \cdot \left(\int (f+g)^{(p-1)q} \right)^{\frac{1}{q}} + \left(\int g^p \right)^{\frac{1}{p}} \cdot \left(\int (f+g)^{(p-1)q} \right)^{\frac{1}{q}} \\ &= \left[\left(\int f^p \right)^{\frac{1}{p}} + \left(\int g^p \right)^{\frac{1}{p}} \right] \cdot \left(\int (f+g)^p \right)^{\frac{p-1}{p}} \end{aligned}$$

Dividing through by $\left(\int (f+g)^p \right)^{\frac{p-1}{p}}$ gives Minkowski's inequality. ///