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05b. L^p spaces, convexity, basic inequalities

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1. Examples: spaces L^p
2. Convexity and inequalities

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1. Examples: spaces L^p

Given a measure space X, for $1 \le p < \infty$ the usual L^p spaces are

$$L^p(X) = \{\text{measurable } f : |f|_{L^p} < \infty\} \text{ modulo } \sim$$

with the usual L^p norm

$$|f|_{L^p} \;=\; \left(\int_X |f|^p\right)^{1/p}$$

and associated metric

$$d(f,g) \;=\; |f-g|_{L^p}$$

taking the quotient by the equivalence relation

$$f \sim g \text{ if } f - g = 0 \text{ off a set of measure } 0$$

[1.1] Remark: These L^p functions have inevitably ambiguous pointwise values, in conflict with the naive formal definition of *function*.

A simple instance of this construction is

$$\ell^p \;=\; \{\text{complex sequences } \{c_i\} \text{ with } \sum_i |c_i|^p < \infty\}$$

with norm $|(c_1, c_2, \ldots)|_{\ell^p} = (\sum_i |c_i|^p)^{1/p}$. The analogue of the following theorem for ℓ^p is more elementary.

[1.2] Theorem: The space $L^p(X)$ is a complete metric space.

[1.3] Remark: In fact, as used in the proof, a Cauchy sequence f_i in $L^p(X)$ has a subsequence converging *pointwise* off a set of measure 0 in X.

Proof: The triangle inequality here is *Minkowski's inequality*. To prove completeness, choose a subsequence f_{n_i} such that

$$|f_{n_i} - f_{n_{i+1}}|_p \;<\; 2^{-i}$$

and put

$$g_n(x) \;=\; \sum_{1 \le i \le n} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and

$$g(x) \;=\; \sum_{1 \le i < \infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

The infinite sum is not necessarily claimed to converge to a finite value for every x. The triangle inequality shows that $|g_n|_p \le 1$. *Fatou's Lemma* asserts that for $[0,\infty]$-valued measurable functions h_i

$$\int_X \left(\liminf_i h_i\right) \;\le\; \liminf_i \int_X h_i$$

Thus, $|g|_p \le 1$, so is finite. Thus,

$$f_{n_1}(x) + \sum_{i \ge 1} \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right)$$

converges for almost all $x \in X$. Let $f(x)$ be the sum at points x where the series converges, and on the measure-zero set where the series does not converge put $f(x) = 0$. Certainly

$$f(x) \;=\; \lim_i f_{n_i}(x) \quad \text{(for almost all } x\text{)}$$

Now prove that this almost-everywhere pointwise limit is the L^p-limit of the original sequence. For $\varepsilon > 0$ take N such that $|f_m - f_n|_p < \varepsilon$ for $m, n \ge N$. Fatou's lemma gives

$$\int |f - f_n|^p \le \liminf_i \int |f_{n_i} - f_n|^p \le \varepsilon^p$$

Thus $f - f_n$ is in L^p and hence f is in L^p. And $|f - f_n|_p \to 0$. ///

[1.4] Theorem: For a locally compact Hausdorff topological space X with positive regular Borel measure μ, the space $C_c^0(X)$ of compactly-supported continuous functions is *dense* in $L^p(X, \mu)$.

Proof: From the definition of *integral* attached to a measure, an L^p function is approximable in L^p metric by a *simple* function, that is, a measurable function assuming only finitely-many values. That is, a simple function is a *finite* linear combination of characteristic functions of measurable sets E. Thus, it suffices to approximate characteristic functions of measurable sets by continuous functions. The assumed *regularity* of the measure gives compact K and open U such that $K \subset E \subset U$ and $\mu(U - E) < \varepsilon$, for given $\varepsilon > 0$. Urysohn's lemma says that there is continuous f identically 1 on K and identically 0 off U. Thus, f approximates the characteristic function of E. ///

[1.5] Corollary: For locally compact Hausdorff X with regular Borel measure μ, $L^p(X, \mu)$ is the L^p-metric completion of $C_c^o(X)$, the compactly-supported continuous functions. ///

[1.6] Remark: *Defining* $L^p(X, \mu)$ to be the L^p completion of $C_c^o(X)$ avoids discussion of ambiguous values on sets of measure zero.

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2. *Convexity and inequalities*

A function f on an interval $(a, b) \subset \mathbb{R}$ is *convex* when its graph bends upward, in the sense that a line segment connecting two points on the graph lies *above* the graph. That is,

$$f(tx + (1-t)y) \;\ge\; tf(x) + (1-t)f(y) \qquad \text{(for } 0 \le t \le 1 \text{ and } a < x < y < b\text{)}$$

The prototype is the exponential function $x \to e^x$.

[2.1] Claim: Convex $\mathbb{R}$-valued functions on an open interval (a, b) (allowing $a = -\infty$ and/or $b = +\infty$) are *continuous.*

Proof: Let g be continuous on (a, b) and take $x \in (a, b)$. Fix any s, t such that $a < s < x < t < b$. For y in the range $x < y < t$, the point $(y, g(y))$ is on or above the line through $(s, g(s))$ and $(x, g(x))$, and is below the line through $(x, g(x))$ and $(t, g(t))$, so $g(y) \to g(x)$ as $y \to x^+$. For $s < y < x$, the same argument gives *left*-continuity. ///

[2.2] Theorem: *(Jensen's inequality)* Let X be a measure space with positive measure of total measure 1. Let $f \in L^1(X)$ be an $\mathbb{R}$-valued function on X with $a < f(x) < b$ for all $x \in X$, where a, b can also be $-\infty$ and $+\infty$. For convex g on (a, b),

$$g\Big(\int_X f\Big) \;\le\; \int_X g \cdot f$$

Proof: First, $a < f(x) < b$ gives $a < \int_X f < b$. The convexity condition can be rewritten as the condition that slopes of secants increase from left to right. Thus, for example,

$$\frac{g(y)-g(x)}{y-x} \;\le\; \frac{g(z)-g(y)}{z-y} \qquad\qquad (\text{for } x<y<z \text{ inside } (a,b))$$

Applying this with $y = \int_X f$,

$$\frac{g(\int f)-g(x)}{\int f - x} \;\le\; \frac{g(z)-g(\int f)}{z-\int f} \qquad\qquad (\text{for all } a<x<\textstyle\int_X f \text{ and for all } \int_X f < z < b)$$

With

$$S \;=\; \sup_{a<x<\int f} \frac{g(\int f)-g(x)}{\int f - x}$$

we have

$$\frac{g(\int f)-g(x)}{\int f - x} \;\le\; S \;\le\; \frac{g(z)-g(\int f)}{z-\int f} \qquad\qquad (\text{for all } a<x<\textstyle\int_X f \text{ and for all } \int_X f < z < b)$$

Thus, from the left half of the latter inequality,

$$g(x) \;\ge\; g(\int_X f) + S\cdot(x - \int_X f) \qquad\qquad (\text{for } a<x<\textstyle\int_X f)$$

and from the right half

$$g(z) \;\ge\; g(\int_X f) + S\cdot(z - \int_X f) \qquad\qquad (\text{for } \textstyle\int_X f < z < b)$$

Thus,

$$g(w) \;\ge\; g(\int_X f) + S\cdot(w - \int_X f) \qquad\qquad (\text{for } \textit{all } w \text{ in the range } a<w<b)$$

In particular, letting $w = f(x)$ now with $x \in X$,

$$g(f(x)) \;\ge\; g(\int_X f) + S\cdot(f(x) - \int_X f) \qquad\qquad (\text{for } \textit{all } w \text{ in the range } a<w<b)$$

Since the convex function g is *continuous*, $g \circ f$ is measurable. Integrating in $x \in X$, using the fact that the total measure is 1,

$$\int_X g\circ f \;\ge\; g(\int_X f) + S\cdot(\int_X f - \int_X f) \;=\; g(\int_X f) + S\cdot 0$$

as claimed. ///

[2.3] Corollary: *(Arithmetic-geometric mean inequality)* For positive real numbers $a_1, \ldots, a_n$,

$$(a_1 a_2 \ldots a_n)^{1/n} \;\le\; \frac{a_1 + a_2 + \ldots + a_n}{n}$$

Proof: In Jensen's inequality, take $g(x) = e^x$, take X a finite set with n (distinct) elements $\{x_1, \ldots, x_n\}$, with each point having measure $1/n$, and $f(x_i) = \log a_i$. Jensen's inequality gives

$$\exp\Big(\frac{\log a_1 + \ldots + \log a_n}{n}\Big) \;\le\; \frac{e^{\log a_1} + \ldots + e^{\log a_n}}{n}$$

which gives the assertion. ///

Conjugate exponents are numbers $p, q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} \;=\; 1$$

For example, p and $\frac{p}{p-1}$ are conjugate exponents.

Generalizing the Cauchy-Schwarz-Bunyakowsky inequality,

[2.4] Corollary: *(Hölder)* For conjugate exponents p, q and $[0, +\infty]$-valued measurable functions f, g,

$$\int_X f \cdot g \;\le\; \Big(\int_X f^p\Big)^{\frac{1}{p}} \cdot \Big(\int_X g^q\Big)^{\frac{1}{q}}$$

Proof: The assertion is trivial if either integral on the right-hand side is $+\infty$ or 0, so suppose the two quantities

$$I \;=\; \Big(\int_X f^p\Big)^{\frac{1}{p}} \qquad\qquad J \;=\; \Big(\int_X g^q\Big)^{\frac{1}{q}}$$

are finite and non-zero. Renormalize by taking $\varphi = f/I$ and $\psi = g/J$, so that $\int \varphi^p = 1 = \int \psi^q$. For $x \in X$ with $0 < \varphi(x) < \infty$ and $0 < \psi(x) < \infty$, there are real numbers u, v such that $e^{u/p} = \varphi(x)$ and $e^{v/q} = \psi(x)$. Invoking Jensen's inequality on a measure space with just two points with measures $\frac{1}{p}$ and $\frac{1}{q}$, using the convexity of the exponential function,

$$\varphi(x)\psi(x) \;=\; e^{\frac{u}{p}+\frac{v}{q}} \;\le\; \frac{e^u}{p} + \frac{e^v}{q} \;=\; \frac{\varphi(x)^p}{p} + \frac{\psi(x)^q}{q}$$

Integrating,

$$\int_X \varphi \cdot \psi \;\le\; \int_X \frac{\varphi(x)^p}{p} + \frac{\psi(x)^q}{q} \;=\; \frac{1}{p} + \frac{1}{q} \;=\; 1$$

From the renormalization, we are done. ///

For the triangle inequality in L^p spaces for general p, we need

[2.5] Corollary: *(Minkowski)* For $1 < p < +\infty$ and $[0, +\infty]$-valued measurable functions f, g,

$$\Big(\int_X (f+g)^p\Big)^{\frac{1}{p}} \;\le\; \Big(\int_X f^p\Big)^{\frac{1}{p}} + \Big(\int_X g^p\Big)^{\frac{1}{p}}$$

Proof: We prove Minkowski's inequality from Hölder's, using the conjugate exponents p and $q = \frac{p}{p-1}$.

$$\int (f+g)^p \;=\; \int f \cdot (f+g)^{p-1} + \int g \cdot (f+g)^{p-1}$$

$$\le\; \Big(\int f^p\Big)^{\frac{1}{p}} \cdot \Big(\int (f+g)^{(p-1)q}\Big)^{\frac{1}{q}} \;+\; \Big(\int g^p\Big)^{\frac{1}{p}} \cdot \Big(\int (f+g)^{(p-1)q}\Big)^{\frac{1}{q}}$$

$$=\; \Big[\Big(\int f^p\Big)^{\frac{1}{p}} + \Big(\int g^p\Big)^{\frac{1}{p}}\Big] \cdot \Big(\int (f+g)^p\Big)^{\frac{p-1}{p}}$$

Dividing through by $\big(\int (f+g)^p\big)^{\frac{p-1}{p}}$ gives Minkowski's inequality. ///